Self-dual projective algebraic varieties associated with symmetric spaces♣

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Abstract

We discover a class of projective self-dual algebraic varieties. Namely, we consider actions of isotropy groups of complex symmetric spaces on the projectivized nilpotent varieties of isotropy modules. For them, we classify all orbit closures X such that $X = \check{X}$ where $\check{X}$ is the projective dual of X. We give algebraic criteria of projective self-duality for the considered orbit closures.

1. Introduction

Under different guises dual varieties of projective algebraic varieties have been considered in various branches of mathematics for over a hundred years. In fact, the dual variety is the generalization to algebraic geometry of the Legendre transform in classical mechanics, and the Biduality Theorem essentially rephrases the duality between the Lagrange and Hamilton–Jacobi approaches in classical mechanics.

Let X be an n-dimensional projective subvariety of an N-dimensional projective space $\mathbf{P}$, and let $\check{X}$ be the dual variety of X in the dual projective space $\check{\mathbf{P}}$. Since various kinds of geometrically meaningful unusual behavior of hyperplane sections are manifested more explicitly in terms of dual varieties, it makes sense to consider their natural invariants. The simplest invariant of $\check{X}$ is its dimension $\check{n}$. "Typically", $\check{n} = N-1$, i.e., $\check{X}$ is a hypersurface. The deviation from the "typical" behavior admits a geometric interpretation: if $\check{X}$ is not a hypersurface and, say, $\operatorname{codim}\check{X} = s+1$, then X is uniruled by s-planes.

Assume that X is a smooth variety not contained in a hyperplane. Then $n \leqslant \check{n}$, by [Z, Chapter 1]. If the extremal case $n = \check{n}$ holds and $n = \check{n} \leqslant 2N/3$, then, by [Ei1], [Ei2], such X are classified by the following list:

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(i) hypersurfaces,
(ii) $\mathbf{P}^1 \times \mathbf{P}^{n-1}$ embedded in $\mathbf{P}^{2n-1}$ by the Segre embedding,
(iii) the Grassmannian of lines in $\mathbf{P}^4$ embedded in $\mathbf{P}^9$ by the Plücker map, or
(iv) the 10-dimensional spinor variety of 4-dimensional linear subspaces on a nonsingular 8-dimensional quadric in $\mathbf{P}^{15}$.

According to Hartshorne's famous conjecture, if $n > 2N/3$, then X is a complete intersection, and hence, by [Ei2], $\check{X}$ is a hypersurface. Therefore it is plausible that the above-stated list contains every smooth X such that $n = \check{n}$. Furthermore, in cases (ii), (iii) and (iv), the variety X is *self-dual* in the sense that, as an embedded variety, $\check{X}$ is isomorphic to X. In case (i), it is self-dual if and only if it is a quadric. Thus, modulo Hartshorne's conjecture, the above-stated list gives the complete classification of all *smooth* self-dual projective algebraic varieties. In particular it shows that there are not many of them.

In [P2], it was found a method for constructing many *nonsmooth* self-dual projective algebraic varieties. It is related to algebraic transformation groups theory: the self-dual projective algebraic varieties appearing in this construction are certain projectivized orbit closures of some linear actions of reductive algebraic groups. We recall this construction.

Let G be a connected reductive algebraic group, let V be a finite dimensional algebraic G-module and let B be a nondegenerate symmetric G-invariant bilinear form on V. We assume that the ground field is $\mathbb{C}$. For a subset S of V, put $S^{\perp} := \{v \in V \mid B(v,s) = 0\ \forall\, s \in S\}$. We identify V and V^* by means of B and denote by $\mathbf{P}$ the associated projective space of $V = V^*$. Thereby the projective dual $\check{X}$ of a Zariski closed irreducible subset X of $\mathbf{P}$ is a Zariski closed subset of $\mathbf{P}$ as well. Let $\mathcal{N}(V)$ be the *null-cone* of V, i.e.,

$$\mathcal{N}(V) := \{v \in V \mid 0 \in \overline{G \cdot v}\},$$

where bar stands for Zariski closure in V.

Theorem 1. ([P2, Theorem 1]) *Assume that there are only finitely many G-orbits in $\mathcal{N}(V)$. Let $v \in \mathcal{N}(V)$ be a nonzero vector and let $X := \mathbf{P}(\overline{G \cdot v}) \subseteq \mathbf{P}$ be the projectivization of its orbit closure. Then the following properties are equivalent:*

(i) $X = \check{X}$.
(ii) $(\mathrm{Lie}(G) \cdot v)^{\perp} \subseteq \mathcal{N}(V)$.

Among the modules covered by this method, there are two naturally allocated classes, namely, that of the adjoint modules and that of the isotropy modules of symmetric spaces. In fact the first class is a subclass of the second one; this subclass has especially nice geometric properties and is studied in more details. Projective self-dual algebraic varieties associated with the adjoint modules by means of Theorem 1 were explicitly classified and studied in [P2]. In particular, according to [P2], these varieties are precisely the projectivized orbit closures of nilpotent elements in the Lie algebra of G that are *distinguished* in the sense of Bala and Carter (see below Theorem 2). This purely algebraic notion plays an important role in the Bala–Carter classification of nilpotent elements, [BaCa].

For the isotropy modules of symmetric spaces, in [P2] it was introduced the notion of (-1)-*distinguished element* of a $\mathbb{Z}_2$-graded semisimple Lie algebra and shown that the projective self-dual algebraic varieties associated with such modules by means of Theorem 1

are precisely the projectivized orbit closures of (-1)-distinguished elements. Thereby classification of such varieties was reduced to the problem of classifying (-1)-distinguished elements. In [P2], it was announced that the latter problem will be addressed in a separate publication.

The goal of the present paper is to give the announced classification: here we explicitly classify (-1)-distinguished elements of all $\mathbb{Z}_2$-graded complex semisimple Lie algebras; this yields the classification of all projective self-dual algebraic varieties associated with the isotropy modules of symmetric spaces by means of the above-stated construction from [P2]. In the last section we briefly discuss some geometric properties of these varieties.

Notice that there are examples of singular projective self-dual algebraic varieties constructed in a different way. For instance, the Kummer surface in $\mathbf{P}^3$, see [GH], or the Coble quartic hypersurface in $\mathbf{P}^7$, see [Pa], are projective self-dual. A series of examples is given by means of "Pyasetskii pairing", [T] (e.g., projectivization of the cone of $n \times m$-matrices of rank $\leqslant n/2$ is projective self-dual for $n \leqslant m$, n even).

Given a projective variety, in general it may be difficult to explicitly identify its dual variety. So our classification contributes to the problem of finding projective varieties X for which $\check{X}$ can be explicitly identified. Another application concerns the problem of explicit describing the projective dual varieties of the projectivized nilpotent orbit closures in the isotropy modules of symmetric spaces: our classification yields its solution for all (-1)-distinguished orbits. To the best of our knowledge, at this writing this problem is largely open; even finding the dimensions of these projective dual varieties would be interesting (for the minimal nilpotent orbit closures in the adjoint modules, a solution was found in [KM]; cf. [Sn] for a short conceptual proof).

2. Main reductions

Let $\mathfrak{g}$ be a semisimple complex Lie algebra, let G be the adjoint group of $\mathfrak{g}$, and let $\theta \in \operatorname{Aut} \mathfrak{g}$ be an element of order 2. We set

$$\mathfrak{k} := \{x \in \mathfrak{g} \mid \theta(x) = x\}, \quad \mathfrak{p} := \{x \in \mathfrak{g} \mid \theta(x) = -x\}. \tag{1}$$

Then $\mathfrak{k}$ and $\mathfrak{p} \neq 0$, the subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is reductive, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a $\mathbb{Z}_2$-grading of the Lie algebra $\mathfrak{g}$, cf., e.g., [OV]. Denote by G the adjoint group of $\mathfrak{g}$. Let K be the connected reductive subgroup of G with the Lie algebra $\mathfrak{k}$; it is the adjoint group of $\mathfrak{k}$.

Consider the adjoint G-module $\mathfrak{g}$. Its null-cone $\mathcal{N}(\mathfrak{g})$ is the cone of all nilpotent elements of $\mathfrak{g}$; it contains only finitely many G-orbits, [D], [K1]. The space $\mathfrak{p}$ is K-stable, and we have

$$\mathcal{N}(\mathfrak{p}) = \mathcal{N}(\mathfrak{g}) \cap \mathfrak{p}.$$

By [KR], there are only finitely many K-orbits in $\mathcal{N}(\mathfrak{p})$ and

$$2 \dim_{\mathbb{C}} K \cdot x = \dim_{\mathbb{C}} G \cdot x \text{ for any } x \in \mathcal{N}(\mathfrak{p}). \tag{2}$$

The Killing form $B_{\mathfrak{g}}$ of $\mathfrak{g}$ (resp., its restriction $B_{\mathfrak{g}}|_{\mathfrak{p}}$ to $\mathfrak{p}$) is symmetric nondegenerate and G-invariant (resp., K-invariant). We identify the linear spaces $\mathfrak{g}$ and $\mathfrak{g}^*$ (resp., $\mathfrak{p}$ and $\mathfrak{p}^*$) by means of $B_{\mathfrak{g}}$ (resp., $B_{\mathfrak{g}}|_{\mathfrak{p}}$).

Recall from [BaCa], [Ca] that a nilpotent element $x\in\mathcal{N}(\mathfrak{g})$ or its G-orbit $G\cdot x$ is called *distinguished* if the centralizer $\mathfrak{z}_{\mathfrak{g}}(x)$ of x in $\mathfrak{g}$ contains no nonzero semisimple elements. According to [P2], distinguished orbits admit the following geometric characterization:

Theorem 2. ([P2, Theorem 2]) *Let $\mathcal{O}$ be a nonzero G-orbit in $\mathcal{N}(\mathfrak{g})$ and $X := \mathbf{P}(\overline{\mathcal{O}})$. The following properties are equivalent:*

(i) *$X = \check{X}$,*
(ii) *$\mathcal{O}$ is distinguished.*

Hence the classification of distinguished elements obtained in [BaCa] yields the classification of projective self-dual orbit closures in $\mathbf{P}(\mathcal{N}(\mathfrak{g}))$. This, in turn, helps studying geometric properties of these projective self-dual projective varieties, [P2].

There is a counterpart of Theorem 2 for the action of K on $\mathfrak{p}$. Namely, if $x\in\mathfrak{p}$, then $\mathfrak{z}_{\mathfrak{g}}(x)$ is θ-stable, hence it is a graded Lie subalgebra of $\mathfrak{g}$,

$$\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}_{\mathfrak{k}}(x)\oplus\mathfrak{z}_{\mathfrak{p}}(x), \quad \text{where} \quad \mathfrak{z}_{\mathfrak{k}}(x) := \mathfrak{z}_{\mathfrak{g}}(x)\cap\mathfrak{k}, \quad \mathfrak{z}_{\mathfrak{p}}(x) := \mathfrak{z}_{\mathfrak{g}}(x)\cap\mathfrak{p}.$$

Definition 1. ([P2]) An element $x\in\mathcal{N}(\mathfrak{p})$ and its K-orbit are called (-1)-*distinguished* if $\mathfrak{z}_{\mathfrak{p}}(x)$ contains no nonzero semisimple elements.

Remark. It is tempting to use simpler terminology and replace "(-1)-distinguished" with merely "distinguished". However this might lead to the ambiguity since, as it is shown below, there are the cases where x is (-1)-distinguished with respect to the action of K on $\mathfrak{p}$ but x is not distinguished with respect to the action of G on $\mathfrak{g}$.

Notice that since $\mathfrak{p}$ contains nonzero semisimple elements (see below the proof of Theorem 7), every (-1)-distinguished element is nonzero.

Theorem 3. ([P2, Theorem 5]) *Let $\mathcal{O}$ be a nonzero K-orbit in $\mathcal{N}(\mathfrak{p})$ and $X := \mathbf{P}(\overline{\mathcal{O}})$. The following properties are equivalent:*

(i) *$X = \check{X}$.*
(ii) *$\mathcal{O}$ is (-1)-distinguished.*

By virtue of Theorem 3, our goal is classifying (-1)-distinguished orbits.

Theorem 4. *Let $\mathcal{O}$ be a K-orbit in $\mathcal{N}(\mathfrak{p})$ and let x be an element of $\mathcal{O}$. The following properties are equivalent:*

(i) *$\mathcal{O}$ is (-1)-distinguished.*
(ii) *The reductive Levi factors of $\mathfrak{z}_{\mathfrak{k}}(x)$ and $\mathfrak{z}_{\mathfrak{g}}(x)$ have the same dimension.*
(iii) *The reductive Levi factors of $\mathfrak{z}_{\mathfrak{k}}(x)$ and $\mathfrak{z}_{\mathfrak{g}}(x)$ are isomorphic.*

If $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{h}$ where $\mathfrak{h}$ is a semisimple complex Lie algebra, and $\theta((y,z)) = (z,y)$, then

$$\mathfrak{k} = \{(y,y) \mid y\in\mathfrak{h}\}, \quad \mathfrak{p} = \{(y,-y) \mid y\in\mathfrak{h}\}. \tag{3}$$

If H is the adjoint group of $\mathfrak{h}$, then (3) implies that K is isomorphic to H and the K-module $\mathfrak{p}$ is isomorphic to the adjoint H-module $\mathfrak{h}$. Therefore, for $y\in\mathcal{N}(\mathfrak{h})$, the variety $\mathbf{P}(\overline{K\cdot y})$ is projective self-dual if and only if the variety $\mathbf{P}(\overline{H\cdot x})$ for $x := (y,-y)\in\mathcal{N}(\mathfrak{p})$ is projective

self-dual. This is consistent with Theorems 2 and 3: the definitions clearly imply that x is (-1)-distinguished if and only y is distinguished. So Theorem 2 follows from Theorem 3.

In [N], it was made an attempt to develop an analogue of the Bala–Carter theory for nilpotent orbits in real semisimple Lie algebras. The Kostant–Sekiguchi bijection (recalled below in this section) reduces this to finding an analogue of the Bala–Carter theory for K-orbits in $\mathcal{N}(\mathfrak{p})$. Given the above discussion, we believe that if such a natural analogue exists, (-1)-distinguished elements should play a key role in it, analogous to that of distinguished elements in the original Bala–Carter theory (in [N], the so called noticed elements, different from (-1)-distinguished ones, play a central role).

Theorems 3, 4 and their proofs are valid over any algebraically closed ground field of characteristic zero. They give a key to explicit classifying (-1)-distinguished elements since there is a method (see the end of this section) for finding reductive Levi factors of $\mathfrak{z}_{\mathfrak{k}}(x)$ and $\mathfrak{z}_{\mathfrak{g}}(x)$.

However, over the complex numbers, there is another nice characterization of (-1)-distinguished orbits. It is given below in Theorem 5. For classical $\mathfrak{g}$, our approach to explicit classifying (-1)-distinguished orbits is based on this result. The very formulation of this characterization uses complex topology. However remark that, by the Lefschetz's principle, the final explicit classification of (-1)-distinguished orbits for classical $\mathfrak{g}$ given by combining Theorem 5 with Theorems 8–13 is valid over any algebraically closed ground field of characteristic zero.

Namely, according to the classical theory (cf., e.g., [OV]), there is a θ-stable real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$ such that

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}, \quad \text{where} \quad \mathfrak{k}_{\mathbb{R}} := \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{k},\ \ \mathfrak{p}_{\mathbb{R}} := \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{p}, \tag{4}$$

is a Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$. The semisimple real algebra $\mathfrak{g}_{\mathbb{R}}$ is noncompact since $\mathfrak{p} \neq 0$. Assigning $\mathfrak{g}_{\mathbb{R}}$ to θ induces well defined bijection from the set of all conjugacy classes of elements of order 2 in $\operatorname{Aut}\mathfrak{g}$ to the set of conjugacy classes of noncompact real forms of $\mathfrak{g}$. If x is an element of $\mathfrak{g}_{\mathbb{R}}$, we put

$$\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(x) := \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{z}_{\mathfrak{g}}(x);$$

$\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(x)$ is the centralizer of x in $\mathfrak{g}_{\mathbb{R}}$. The identity component of the Lie group of real points of G is the adjoint group $\operatorname{Ad}(\mathfrak{g}_{\mathbb{R}})$ of $\mathfrak{g}_{\mathbb{R}}$. We set

$$\mathcal{N}(\mathfrak{g}_{\mathbb{R}}) := \mathcal{N}(\mathfrak{g}) \cap \mathfrak{g}_{\mathbb{R}}.$$

Definition 2. An element $x \in \mathcal{N}(\mathfrak{g}_{\mathbb{R}})$ is called *compact* if the reductive Levi factor of its centralizer $\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(x)$ is a compact Lie algebra.

Recall that there is a bijection between the sets of nonzero K-orbits in $\mathcal{N}(\mathfrak{p})$ and nonzero $\operatorname{Ad}(\mathfrak{g}_{\mathbb{R}})$-orbits in $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})$, cf. [CM], [M]. Namely, let σ be the complex conjugation of $\mathfrak{g}$ defined by $\mathfrak{g}_{\mathbb{R}}$, viz.,

$$\sigma(a + ib) = a - ib,\ \ a, b \in \mathfrak{g}_{\mathbb{R}}.$$

Let $\{e, h, f\}$ be an $\mathfrak{sl}_2$*-triple* in $\mathfrak{g}$, i.e., an ordered triple of elements of $\mathfrak{g}$ spanning a three-dimensional simple subalgebra of $\mathfrak{g}$ and satisfying the bracket relations

$$[h, e] = 2e,\ \ [h, f] = -2f,\ \ [e, f] = h. \tag{5}$$

It is called a *complex Cayley triple* if $e, f \in \mathfrak{p}$ (hence $h \in \mathfrak{k}$) and $\sigma(e) = -f$. Given a complex Cayley triple $\{e, h, f\}$, set

$$e' := i(-h + e + f)/2, \quad h' := e - f, \quad f' := -i(h + e + f)/2. \tag{6}$$

Then $\{e', h', f'\}$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_{\mathbb{R}}$ such that $\theta(e') = f'$ (and hence $\theta(f') = e', \theta(h') = -h'$). The $\mathfrak{sl}_2$-triples in $\mathfrak{g}_{\mathbb{R}}$ satisfying the last property are called the *real Cayley triples*. Given $\{e', h', f'\}$, the following formulas restore $\{e, h, f\}$:

$$e := (h' + if' - ie')/2, \quad h := i(e' + f'), \quad f := (-h' + if' - ie')/2. \tag{7}$$

The map $\{e, h, f\} \mapsto \{e', h', f'\}$ is a bijection from the set of complex to the set of real Cayley triples. The triple $\{e, h, f\}$ is called the *Cayley transform* of $\{e', h', f'\}$.

Now let $\mathcal{O}$ be a nonzero K-orbit in $\mathcal{N}(\mathfrak{p})$. Then, according to [KR], there exists a complex Cayley triple $\{e, h, f\}$ in $\mathfrak{g}$ such that $e \in \mathcal{O}$. Let $\{e', h', f'\}$ be the real Cayley triple in $\mathfrak{g}_{\mathbb{R}}$ such that $\{e, h, f\}$ is its Cayley transform. Let $\mathcal{O}' = \mathrm{Ad}(\mathfrak{g}_{\mathbb{R}}) \cdot e'$. Then the map assigning $\mathcal{O}'$ to $\mathcal{O}$ is well defined and establishes a bijection, called the *Kostant–Sekiguchi bijection*, between the set of nonzero K-orbits in $\mathcal{N}(\mathfrak{p})$ and the set of nonzero $\mathrm{Ad}(\mathfrak{g}_{\mathbb{R}})$-orbits in $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})$. We say that the orbits $\mathcal{O}$ and $\mathcal{O}'$ *correspond* one another via the Kostant–Sekiguchi bijection. One can show, [KR], cf. [CM], that

$$\begin{gathered} G \cdot e' = G \cdot e, \\ \dim_{\mathbb{R}} \mathrm{Ad}(\mathfrak{g}_{\mathbb{R}}) \cdot e' = \dim_{\mathbb{C}} G \cdot e'. \end{gathered} \tag{8}$$

Theorem 5. *Let $\mathcal{O}$ be a nonzero K-orbit in $\mathcal{N}(\mathfrak{p})$ and let x be an element of the* $\mathrm{Ad}(\mathfrak{g}_{\mathbb{R}})$*-orbit in $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})$ corresponding to $\mathcal{O}$ via the Kostant–Sekiguchi bijection. The following properties are equivalent:*

(i) *$\mathcal{O}$ is (-1)-distinguished.*
(ii) *x is compact.*

Next result, Theorem 6, reduces studying projective self-dual varieties associated with symmetric spaces to the case of simple Lie algebras $\mathfrak{g}$. Namely, since the Lie algebra $\mathfrak{g}$ is semisimple, it is the direct sum of all its simple ideals. As this set of ideals is θ-stable, $\mathfrak{g}$ is the direct sum of its θ-stable ideals, $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_d$, where each $\mathfrak{g}_l$ is either

(a) simple or
(b) direct sum of two isomorphic simple ideals permuted by θ.

Let G_l, $\mathfrak{k}_l$, $\mathfrak{p}_l$, K_l and $\mathcal{N}(\mathfrak{p}_l)$ have the same meaning for $\mathfrak{g}_l$ with respect to $\theta_l := \theta|_{\mathfrak{g}_l}$ as resp., G, $\mathfrak{k}$, $\mathfrak{p}$, K and $\mathcal{N}(\mathfrak{p})$ have for $\mathfrak{g}$ with respect to θ. Then $G = G_1 \times \ldots \times G_d$, $\mathfrak{k} = \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_d$, $\mathfrak{p} = \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_d$, $K = K_1 \times \ldots \times K_d$ and $\mathcal{N}(\mathfrak{p}) = \mathcal{N}(\mathfrak{p}_1) \times \ldots \times \mathcal{N}(\mathfrak{p}_d)$.

In Theorem 6, we use the following notation. Let $X_1, \ldots, X_s$ be the closed subvarieties of a projective space $\mathbf{P}$, and let $s \geqslant 2$. Consider the variety

$$\overline{\{(x_1, \ldots, x_s, y) \in X_1 \times \ldots \times X_s \times \mathbf{P} \mid \dim\langle x_1, \ldots, x_s\rangle = s - 1,\ y \in \langle x_1, \ldots, x_s\rangle\}}, \tag{9}$$

where bar denotes Zariski closure in $X_1 \times \ldots \times X_s \times \mathbf{P}$ and $\langle S\rangle$ denotes the linear span of S in $\mathbf{P}$. Consider the projection of variety (9) to $\mathbf{P}$. Its image is denoted by

$$\mathrm{Join}(X_1, \ldots, X_s) \tag{10}$$

and called the *join* of $X_1, \ldots, X_s$. If $s = 2$, then (10) is the usual join of X_1 and X_2, cf. [Ha]. If $s > 2$, then (10) is the usual join of $\mathrm{Join}(X_1, \ldots, X_{s-1})$ and X_s.

Theorem 6. *Let $x = x_1 + \ldots + x_d$ where $x_l \in \mathcal{N}(\mathfrak{p}_l)$, $l = 1, \ldots, d$. Consider in $\mathbf{P}(\mathfrak{g})$ the projective subvarieties $X := \mathbf{P}(\overline{K \cdot x})$ and $X_l := \mathbf{P}(\overline{K_l \cdot x_l})$. Then*

$$X = \mathrm{Join}(X_1, \ldots, X_d)$$

and the following properties are equivalent:

(i) $X = \check{X}$.

(ii) $X_l = \check{X}_l$ *for all* l.

Remark. This is a specific geometric property of the considered varieties. In general setting, projective self-duality of $\mathrm{Join}(Z_1, \ldots, Z_m)$ is not equivalent to projective self-duality of all $Z_1, \ldots, Z_m$.

If $\mathfrak{g}_l$ is of type (b), viz., $\mathfrak{g}_l$ is isomorphic to the direct sum of algebras $\mathfrak{s} \oplus \mathfrak{s}$, where $\mathfrak{s}$ is a simple algebra, and θ_l acts by $\theta_l((x, y)) = (y, x)$, then, as explained above (see (3)), classifying (-1)-distinguished orbits in $\mathcal{N}(\mathfrak{p}_l)$ amounts to classifying distinguished orbits in $\mathcal{N}(\mathfrak{s})$. Since the last classification is known, [BaCa], [Ca], this and Theorem 6 reduce classifying (-1)-distinguished orbits in $\mathfrak{g}$ to the case where $\mathfrak{g}$ is a *simple* Lie algebra.

In this case, for explicit classifying (-1)-distinguished orbits, one can apply Theorem 4 since there are algorithms, [Ka], [Vi], yielding, in principle, a classification of K-orbits in $\mathcal{N}(\mathfrak{p})$ and a description of the reductive Levi factors of $\mathfrak{z}_{\mathfrak{k}}(x)$ and $\mathfrak{z}_{\mathfrak{g}}(x)$ for $x \in \mathcal{N}(\mathfrak{p})$. Moreover, there is a computer program, [L], implementing these algorithms and yielding, for concrete pairs $(\mathfrak{g}, \theta)$, the explicit representatives x of K-orbits in $\mathcal{N}(\mathfrak{p})$ and the reductive Levi factors of $\mathfrak{z}_{\mathfrak{k}}(x)$ and $\mathfrak{z}_{\mathfrak{g}}(x)$. There is also another way for finding representatives and dimensions of K-orbits in $\mathcal{N}(\mathfrak{p})$: they may be obtained using the general algorithm of finding Hesselink strata for linear reductive group actions given in [P3], since in our case these strata coincide with K-orbits, see [P3, Proposition 4].

Fortunately, for exceptional simple $\mathfrak{g}$, explicit finding of the aforementioned classification and description already has been performed by D. Ž. Đoković in [D3], [D4], cf. [CM]. In these papers, the answers are given in terms of the so called *characteristics* in the sense of Dynkin (in Section 5, we recall this classical approach, [D], [K1], [KR], cf. [CM] and [Vi]). Combining either of Theorems 4 and 5 with these results, we obtain, for all exceptional simple $\mathfrak{g}$ and all θ, the explicit classification of (-1)-distinguished K-orbits in $\mathcal{N}(\mathfrak{p})$ in terms of their characteristics (see Theorem 14).

For classical $\mathfrak{g}$, we use another approach and obtain the classification of (-1)-distinguished K-orbits in $\mathcal{N}(\mathfrak{p})$ by means of the Kostant–Sekiguchi bijection, Theorem 5 and elementary representation theory of $\mathfrak{sl}_2$ (see Theorems 8–13).

Theorems 4, 5, 6 are proved in Section 3. For simple Lie algebras $\mathfrak{g}$, the classifications of (-1)-distinguished K-orbits in $\mathcal{N}(\mathfrak{p})$ are obtained in the next two sections: in Section 4, we consider the case of classical $\mathfrak{g}$, and in Section 5, that of exceptional simple $\mathfrak{g}$. In Section 6, we briefly discuss some geometric properties of the constructed self-dual projective algebraic varieties.

3. Proofs of Theorems 4–6

Proof of Theorem 4. Theorem 4 immediately follows from Theorem 7 proved below.

Let k be a field of characteristic 0. If $\mathfrak{c}$ is a finite dimensional algebraic Lie algebra over k, we denote by $\operatorname{rad}\mathfrak{c}$ (resp., $\operatorname{rad}_u\mathfrak{c}$) the radical (resp., the unipotent radical, i.e., the maximal ideal whose elements are nilpotent) of $\mathfrak{c}$. We call $\mathfrak{c}/\operatorname{rad}\mathfrak{c}$ (resp., $\mathfrak{c}/\operatorname{rad}_u\mathfrak{c}$) the *Levi factor* (resp., the *reductive Levi factor*) of $\mathfrak{c}$. According to [Ch], there is a semisimple (resp., reductive) subalgebra $\mathfrak{l}$ in $\mathfrak{c}$ such that $\mathfrak{c}$ is the semidirect sum of $\mathfrak{l}$ and $\operatorname{rad}\mathfrak{c}$ (resp., $\operatorname{rad}_u\mathfrak{c}$). Every such $\mathfrak{l}$ is called *Levi subalgebra* (resp., *reductive Levi subalgebra*) of $\mathfrak{c}$.

Theorem 7. *Assume that k is algebraically closed and let $\mathfrak{a}$ be a finite dimensional algebraic Lie algebra over k. Let $\theta \in \operatorname{Aut}\mathfrak{a}$ be an element of order 2. For any θ-stable linear subspace $\mathfrak{l}$ of $\mathfrak{a}$, set*

$$\mathfrak{l}^{\pm}=\{x\in\mathfrak{l}\mid \theta(x)=\pm x\}.$$

The following properties are equivalent:

(i) $\mathfrak{a}^-$ *contains no nonzero semisimple elements.*

(ii) $\mathfrak{a}^- = (\operatorname{rad}_u\mathfrak{a})^-$.

(iii) $\dim\mathfrak{a}^- = \dim(\operatorname{rad}_u\mathfrak{a})^-$.

(iv) $\mathfrak{a}^- \subseteq \operatorname{rad}_u\mathfrak{a}$.

(v) *The reductive Levi factors of $\mathfrak{a}^+$ and $\mathfrak{a}$ have the same dimension.*

(vi) *The reductive Levi factors of $\mathfrak{a}^+$ and $\mathfrak{a}$ are isomorphic.*

(vii) *The set of all reductive Levi subalgebras of $\mathfrak{a}^+$ coincides with the set of all θ-stable reductive Levi subalgebras of $\mathfrak{a}$.*

Proof. It is known that $\mathfrak{a}$ contains a θ-stable Levi subalgebra $\mathfrak{s}$ (e. g., see [KN, Appendix 9]), and $\operatorname{rad}\mathfrak{a}$ contains a θ-stable maximal torus $\mathfrak{t}$, see [St]. Thus we have the following θ-stable direct sum decompositions of vector spaces:

$$\begin{cases} \mathfrak{a}=\mathfrak{s}\oplus\operatorname{rad}\mathfrak{a}, \quad \operatorname{rad}\mathfrak{a}=\mathfrak{t}\oplus\operatorname{rad}_u\mathfrak{a}, \\ \mathfrak{s}=\mathfrak{s}^+\oplus\mathfrak{s}^-, \quad \mathfrak{t}=\mathfrak{t}^+\oplus\mathfrak{t}^-, \quad \operatorname{rad}_u\mathfrak{a}=(\operatorname{rad}_u\mathfrak{a})^+\oplus(\operatorname{rad}_u\mathfrak{a})^-, \\ \mathfrak{a}^{\pm}=\mathfrak{s}^{\pm}\oplus\mathfrak{t}^{\pm}\oplus(\operatorname{rad}_u\mathfrak{a})^{\pm}. \end{cases} \tag{11}$$

(i) $\Rightarrow$ (ii): Assume that (i) holds. Then, by (11), we have $\mathfrak{t}^-=0$. If $\mathfrak{s}^-\neq 0$, then $\theta|_{\mathfrak{s}}\in\operatorname{Aut}\mathfrak{a}$ is an element of order 2. By [Vu], this implies that $\mathfrak{s}^-$ contains a nonzero θ-stable algebraic torus. By (11), this contradicts (i). Thus $\mathfrak{s}^-=0$. Whence (ii) by (11).

(ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i): This is because all elements of $\operatorname{rad}_u\mathfrak{a}$ are nilpotent.

(ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv): This is evident.

(iii) $\Rightarrow$ (ii): This follows from (11).

(ii) $\Rightarrow$ (v): It follows from (11) that (ii) is equivalent to

$$\mathfrak{s}\oplus\mathfrak{t}\subseteq\mathfrak{a}^+. \tag{12}$$

Assume that (12) holds. Let $\pi:\mathfrak{a}\to\mathfrak{a}/\operatorname{rad}_u\mathfrak{a}$ be the natural homomorphism. By (11), (12), we have $\pi(\mathfrak{a}^+)=\mathfrak{a}/\operatorname{rad}_u\mathfrak{a}$. Since the algebra $\mathfrak{a}/\operatorname{rad}_u\mathfrak{a}$ is reductive, this implies

$$\begin{aligned} \dim\mathfrak{a}/\operatorname{rad}_u\mathfrak{a} &\leqslant \text{dimension of reductive Levi subalgebras of } \mathfrak{a}^+ \\ &= \dim\mathfrak{a}^+/\operatorname{rad}_u(\mathfrak{a}^+). \end{aligned} \tag{13}$$

On the other hand, since reductive Levi subalgebras of $\mathfrak{a}$ and $\mathfrak{a}^+$ are precisely their maximal reductive subalgebras, cf., e.g., [OV] , the inclusion $\mathfrak{a}^+ \subseteq \mathfrak{a}$ implies

$$\begin{aligned} \dim \mathfrak{a}/\mathrm{rad}_u\mathfrak{a} &= \text{dimension of reductive Levi subalgebras of } \mathfrak{a} \\ &\geqslant \text{dimension of reductive Levi subalgebras of } \mathfrak{a}^+. \end{aligned} \tag{14}$$

Now (v) follows from (13), (14).

(v) $\Rightarrow$ (ii): Assume that (v) holds. Let $\mathfrak{l}$ be a reductive Levi subalgebra of $\mathfrak{a}^+$. Since reductive Levi subalgebras of $\mathfrak{a}$ are precisely its maximal reductive subalgebras, it follows from (v) that $\mathfrak{l}$ is a reductive Levi subalgebra of $\mathfrak{a}$ as well. Let $\mathfrak{s}$ and $\mathfrak{t}$ be resp. the derived subalgebra and the center of $\mathfrak{l}$; so we have $\mathfrak{l} = \mathfrak{s} \oplus \mathfrak{t}$. Then $\mathfrak{s}$ is a θ-stable Levi subalgebra of $\mathfrak{a}$, and $\mathfrak{t}$ is a θ-stable maximal torus of $\mathrm{rad}\,\mathfrak{a}$. Hence, as above, (ii) is equivalent to the inclusion (12). As the latter holds by the very construction of $\mathfrak{l}$, we conclude that (ii) holds as well.

(vii) $\Rightarrow$ (v): This is clear.

(i) $\Rightarrow$ (vii): Let $\mathfrak{l}$ be a θ-stable reductive Levi subalgebra of $\mathfrak{a}$. If $\mathfrak{l}^- \neq 0$, then $\theta|_{\mathfrak{l}} \in \mathrm{Aut}\,\mathfrak{l}$ is an element of order 2. Hence, by [Vu], there is a nonzero θ-stable algebraic torus in $\mathfrak{l}^-$. This contradicts (i). Whence $\mathfrak{l}^- = 0$, i.e., $\mathfrak{l}$ lies in $\mathfrak{a}^+$.

(vi) $\Rightarrow$ (v) and (vii) $\Rightarrow$ (vi): This is clear. $\square$

Proof of Theorem 5. According to [KR], there is a complex Cayley triple $\{e, h, f\}$ in $\mathfrak{g}$ such that $\mathcal{O} = K \cdot e$. Let $\{e', h', f'\}$ be the real Cayley triple in $\mathfrak{g}_{\mathbb{R}}$ whose Cayley transform is $\{e, h, f\}$. Since $\mathcal{O}' = \mathrm{Ad}(\mathfrak{g}_{\mathbb{R}}) \cdot e'$, we may, and will, assume that $x = e'$.

Let $\mathfrak{s}$ (resp., $\mathfrak{s}_{\mathbb{R}}$) be the simple three-dimensional subalgebra of $\mathfrak{g}$ (resp., $\mathfrak{g}_{\mathbb{R}}$) spanned by $\{e, h, f\}$ (resp., $\{e', h', f'\}$),

$$\mathfrak{s} = \mathbb{C}e + \mathbb{C}h + \mathbb{C}f, \quad \mathfrak{s}_{\mathbb{R}} = \mathbb{R}e' + \mathbb{R}h' + \mathbb{R}f'. \tag{15}$$

Denote by $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ (resp., $\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}})$) the centralizer of $\mathfrak{s}$ (resp., $\mathfrak{s}_{\mathbb{R}}$) in $\mathfrak{g}$ (resp., $\mathfrak{g}_{\mathbb{R}}$).

It follows from (6), (7), (15) that $\mathfrak{s}_{\mathbb{R}}$ is a real form of $\mathfrak{s}$. Since $\mathfrak{g}_{\mathbb{R}}$ is a real form of $\mathfrak{g}$, this yields that $\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}})$ is a real form of $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$. The definitions of complex and real Cayley triples imply that $\mathfrak{s}$ and $\mathfrak{s}_{\mathbb{R}}$ are θ-stable subalgebras. Therefore $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ and $\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}})$ are θ-stable as well. Whence

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = (\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \cap \mathfrak{k}) \oplus (\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \cap \mathfrak{p}), \quad \mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) = (\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \cap \mathfrak{k}_{\mathbb{R}}) \oplus (\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \cap \mathfrak{p}_{\mathbb{R}}), \tag{16}$$

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \cap \mathfrak{k} = \mathbb{C}(\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \cap \mathfrak{k}_{\mathbb{R}}), \quad \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \cap \mathfrak{p} = \mathbb{C}(\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \cap \mathfrak{p}_{\mathbb{R}}). \tag{17}$$

Now take into account that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ (resp., $\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}})$) is the reductive Levi subalgebra of $\mathfrak{z}_{\mathfrak{g}}(e)$ (resp., $\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(e')$): regarding $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$, see, e.g., [SpSt]; for $\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}})$, the arguments are the same. Hence, by Theorem 7, the orbit $\mathcal{O}$ is (-1)-distinguished if and only if $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) \cap \mathfrak{p} = 0$. By (17), the latter condition is equivalent to $\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \cap \mathfrak{p}_{\mathbb{R}} = 0$. On the other hand, since (4) is the Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$, it follows from (16) that $\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}}) \cap \mathfrak{p}_{\mathbb{R}} = 0$ if and only if the Lie algebra $\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{s}_{\mathbb{R}})$ is compact. This completes the proof of Theorem 5. $\square$

Proof of Theorem 6. Clearly the following proposition implies the first statement.

Proposition 1. ([P2, Proposition 2]) *Let H_t be an algebraic group and let L_t be a finite dimensional algebraic H_t-module, $t = 1, 2$. Let $v_t \in L_s$ be a nonzero vector such that the orbit $H_t \cdot v_t$ is stable with respect to scalar multiplications. Put $H := H_1 \times H_2$, $L := L_1 \oplus L_2$. Identify L_t with the linear subspace of L and set $v := v_1 + v_2$. Then*

$$\mathbf{P}(\overline{H \cdot v}) = \mathrm{Join}(\mathbf{P}(\overline{H_1 \cdot v_1}), \mathbf{P}(\overline{H_2 \cdot v_2})).$$

Regarding the second statement, notice that the centralizers $\mathfrak{z}_{\mathfrak{g}_l}(x_l)$ and $\mathfrak{z}_{\mathfrak{g}}(x)$ are resp. θ_l- and θ-stable and $\mathfrak{z}_{\mathfrak{p}}(x) = \mathfrak{z}_{\mathfrak{p}_1}(x_1) \oplus \ldots \oplus \mathfrak{z}_{\mathfrak{p}_d}(x_d)$. Now the claim follows from this decomposition, Definition 1 and Theorem 3. $\square$

4. Classification for classical $\mathfrak{g}$

If $\mathfrak{g}$ is classical, our approach to classifying (-1)-distinguished nilpotent K-orbits in $\mathfrak{p}$ is based on the Kostant–Sekiguchi bijection and Theorem 5. Namely, for every noncompact real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$, we will find in $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})$ all $\mathrm{Ad}(\mathfrak{g}_{\mathbb{R}})$-orbits $\mathcal{O}$ whose elements are compact. If this is done, then for the complexification of a Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$, the K-orbits in $\mathcal{N}(\mathfrak{p})$ corresponding to all such $\mathcal{O}$ via the Kostant–Sekiguchi bijection are precisely all (-1)-distinguished orbits.

First we recall the classification of real forms $\mathfrak{g}_{\mathbb{R}}$ of classical complex Lie algebras $\mathfrak{g}$. Let D be either real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, or quaternions $\mathbb{H}$. If $a \in D$, we denote by $\overline{a}$ the conjugate of a. Let V be a finite dimensional vector space over D (left D-module). Denote by $\mathrm{GL}_D(V)$ the Lie group of all D-linear automorphisms of V. Its Lie algebra $\mathfrak{gl}_D(V)$ is identified with the real Lie algebra of all D-linear endomorphisms of V. The derived algebra of $\mathfrak{gl}_D(V)$ is denoted by $\mathfrak{sl}_D(V)$.

Let a map $\Phi : V \times V \to D$ be D-linear with respect to the first argument. It is called

— *symmetric bilinear form on V* if $\Phi(x,y) = \Phi(y,x)$ for all $x, y \in V$,
— *skew-symmetric bilinear form on V* if $\Phi(x,y) = -\Phi(y,x)$ for all $x, y \in V$,
— *Hermitian form on V* if $\Phi(x,y) = \overline{\Phi(y,x)}$ for all $x, y \in V$,
— *skew-Hermitian form on V* if $\Phi(x,y) = -\overline{\Phi(y,x)}$ for all $x, y \in V$.

By "form on V" we always mean form of one of these four types. A form Φ on V is called *nondegenerate* if, for any $x \in V$, there is $y \in V$ such that $\Phi(x,y) \neq 0$. If Φ_s, $s = 1, 2$, is a form on a finite dimensional vector space V_s over D, then Φ_1 and Φ_2 are called *equivalent forms* if there exists an isometry $\psi : V_1 \to V_2$ with respect to Φ_1 and Φ_2, i.e., an isomorphism of vector spaces over D such that $\Phi_2(\psi(x), \psi(y)) = \Phi_1(x,y)$ for all $x, y \in V_1$.

The automorphism group of a form Φ on V,

$$\mathrm{GL}_D^{\Phi}(V) := \{g \in \mathrm{GL}_D(V) \mid \Phi(g \cdot x, g \cdot y) = \Phi(x,y) \text{ for all } x, y \in V\},$$

is a Lie subgroup of $\mathrm{GL}_D(V)$ whose Lie algebra is

$$\mathfrak{gl}_D^{\Phi}(V) := \{A \in \mathfrak{gl}_D(V) \mid \Phi(A \cdot x, y) + \Phi(x, A \cdot y) = 0 \text{ for all } x, y \in V\}.$$

For $\Phi = 0$ (zero form), we have $\mathrm{GL}_D^{\Phi}(V) = \mathrm{GL}_D(V)$ and $\mathfrak{gl}_D^{\Phi}(V) = \mathfrak{gl}_D(V)$.

Let $n := \dim_D V$ and let $(e) := \{e_1, \ldots, e_n\}$ be a basis of V over D. Identifying D-linear endomorphisms of V with their matrices with respect to (e), we identify $\mathfrak{gl}_D(V)$, $\mathfrak{sl}_D(V)$ and

$\mathfrak{gl}_D^\Phi(V)$ with the corresponding matrix real Lie algebras. For $D=\mathbb{R}$, $\mathbb{H}$, the real Lie algebras $\mathfrak{gl}_D(V)$ and $\mathfrak{sl}_D(V)$ are denoted resp. by $\mathfrak{gl}_n(D)$ and $\mathfrak{sl}_n(D)$, and for $D=\mathbb{C}$, by $\mathfrak{gl}_n(\mathbb{C})_\mathbb{R}$ and $\mathfrak{sl}_n(\mathbb{C})_\mathbb{R}$. The last algebras are endowed with the natural structures of complex Lie algebras that are denoted resp. by $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{sl}_n(\mathbb{C})$. If $D=\mathbb{H}$, then V is a $2n$-dimensional vector space over the subfield $\mathbb{C}$ of $\mathbb{H}$, and elements of $\mathfrak{gl}_\mathbb{H}(V)$ are its $\mathbb{C}$-linear endomorphisms. Identifying them with their matrices with respect to the basis $e_1,\dots,e_n,je_1,\dots,je_n$, we identify $\mathfrak{gl}_n(\mathbb{H})$ (resp., $\mathfrak{sl}_n(\mathbb{H})$) with the corresponding Lie subalgebra of $\mathfrak{gl}_{2n}(\mathbb{C})_\mathbb{R}$ (resp., $\mathfrak{sl}_{2n}(\mathbb{C})_\mathbb{R}$).

If Φ is a form on V, then the matrix (Φ_{st}) of Φ with respect to (e) defines Φ by

$$\Phi\big(\textstyle\sum_{s=1}^n x_se_s, \sum_{t=1}^n y_te_t\big)=\begin{cases}\sum_{s,t=1}^n x_s\Phi_{st}y_t & \text{if } \Phi \text{ is symmetric or skew-symmetric,}\\ \sum_{s,t=1}^n x_s\Phi_{st}\overline{y_t} & \text{if } \Phi \text{ is Hermitian or skew-Hermitian.}\end{cases}$$

The map $\Phi\mapsto(\Phi_{st})$ is a bijection between all symmetric (resp., skew-symmetric, Hermitian, skew-Hermitian) forms on V and all symmetric (resp., skew-symmetric, Hermitian, skew-Hermitian) $n\times n$-matrices over D. Φ is nondegenerate if and only if (Φ_{st}) is nonsingular.

We denote by I_d the unit matrix of size d and put $\mathrm{I}_{p,q}:=\mathrm{diag}(1,\dots,1,-1,\dots,-1)$ where p (resp., q) is the number of 1's (resp., -1's).

Now we summarize the results from linear algebra about classification of forms and fix some notation and terminology. In the sequel, $\mathbb{N}$ denotes the set of all nonnegative integers.

Symmetric bilinear forms

$D=\mathbb{R}$. There are exactly $n+1$ equivalence classes of nondegenerate symmetric bilinear forms on V. They are represented by the forms Φ with $(\Phi_{st})=\mathrm{I}_{p,q}$, $p+q=n$, $p=0,\dots,n$. If $(\Phi_{st})=\mathrm{I}_{p,q}$, then $\operatorname{sgn}\Phi:=p-q$ is called the *signature of* Φ, and the real Lie algebra $\mathfrak{gl}_D^\Phi(V)$ is denoted by $\mathfrak{so}_{p,q}$.

$D=\mathbb{C}$. There is exactly one equivalence class of nondegenerate symmetric bilinear forms on V. It is represented by the form Φ with $(\Phi_{st})=\mathrm{I}_n$. The corresponding real Lie algebra $\mathfrak{gl}_D^\Phi(V)$ is denoted by $\mathfrak{so}_n(\mathbb{C})_\mathbb{R}$. It has a natural structure of complex Lie algebra that is denoted by $\mathfrak{so}_n(\mathbb{C})$.

$D=\mathbb{H}$. There are no nonzero symmetric bilinear forms on V.

Skew-symmetric bilinear forms

$D=\mathbb{R}$ and $\mathbb{C}$. Nondegenerate skew-symmetric bilinear forms on V exist if and only if n is even. In this case, there is exactly one equivalence class of such forms. It is represented by the form Φ with $(\Phi_{st})=\left[\begin{smallmatrix}0 & \mathrm{I}_{n/2}\\ -\mathrm{I}_{n/2} & 0\end{smallmatrix}\right]$. Corresponding real Lie algebra $\mathfrak{gl}_D^\Phi(V)$ is denoted by $\mathfrak{sp}_n(\mathbb{R})$ for $D=\mathbb{R}$, and by $\mathfrak{sp}_n(\mathbb{C})_\mathbb{R}$ for $D=\mathbb{C}$. The algebra $\mathfrak{sp}_n(\mathbb{C})_\mathbb{R}$ has a natural structure of complex Lie algebra that is denoted by $\mathfrak{sp}_n(\mathbb{C})$.

$D=\mathbb{H}$. There are no nonzero skew-symmetric bilinear forms on V.

Hermitian forms

$D=\mathbb{R}$. Hermitian forms on V coincide with symmetric bilinear forms.

$D=\mathbb{C}$ and $\mathbb{H}$. There are exactly $n+1$ equivalence classes of nondegenerate Hermitian forms on V. They are represented by the forms Φ with $(\Phi_{st})=\mathrm{I}_{p,q}$, $p+q=n$, $p=0,\dots,n$.

If $(\Phi_{st}) = \mathrm{I}_{p,q}$, then $\operatorname{sgn}\Phi := p-q$ is called the *signature* of Φ, and the real Lie algebra $\mathfrak{gl}_D^\Phi(V)$ is denoted by $\mathfrak{su}_{p,q}$ for $D=\mathbb{C}$, and by $\mathfrak{sp}_{p,q}$ for $D=\mathbb{H}$.

Skew-Hermitian forms

$D=\mathbb{R}$. Skew-Hermitian forms on V coincide with skew-symmetric forms.

$D=\mathbb{C}$. The map $\Phi\mapsto i\Phi$ is a bijection between all nondegenerate Hermitian forms on V and all nondegenerate skew-Hermitian forms on V.

$D=\mathbb{H}$. There is exactly one equivalence class of nondegenerate skew-Hermitian forms on V. It is represented by the form Φ with $(\Phi_{st}) = j\mathrm{I}_n$. The corresponding real Lie algebra $\mathfrak{gl}_D^\Phi(V)$ is denoted by $\mathfrak{u}_n^*(\mathbb{H})$.

Some of the above-defined Lie algebras are isomorphic to each other. Obviously $\mathfrak{su}_{p,q} = \mathfrak{su}_{q,p}$, $\mathfrak{so}_{p,q} = \mathfrak{so}_{q,p}$, $\mathfrak{sp}_{p,q} = \mathfrak{sp}_{q,p}$ and we also have

$$\begin{cases} \mathfrak{so}_2 = \mathfrak{u}_1^*(\mathbb{H}),\ \mathfrak{so}_3(\mathbb{C}) \simeq \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sp}_2(\mathbb{C}),\ \mathfrak{so}_3 \simeq \mathfrak{su}_2 = \mathfrak{sp}_{1,0} = \mathfrak{sl}_1(\mathbb{H}), \\ \mathfrak{so}_{1,2} \simeq \mathfrak{su}_{1,1} \simeq \mathfrak{sl}_2(\mathbb{R}) = \mathfrak{sp}_2(\mathbb{R}),\ \mathfrak{so}_4(\mathbb{C}) \simeq \mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{sl}_2(\mathbb{C}),\ \mathfrak{so}_4 \simeq \mathfrak{su}_2\oplus\mathfrak{su}_2, \\ \mathfrak{so}_{1,3} \simeq \mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}, \mathfrak{so}_{2,2} \simeq \mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{sl}_2(\mathbb{R}), \mathfrak{u}_2^*(\mathbb{H}) \simeq \mathfrak{su}_2\oplus\mathfrak{sl}_2(\mathbb{R}), \mathfrak{so}_5(\mathbb{C}) \simeq \mathfrak{sp}_4(\mathbb{C}), \\ \mathfrak{so}_5 \simeq \mathfrak{sp}_2, \mathfrak{so}_{1,4} \simeq \mathfrak{sp}_{1,1},\ \mathfrak{so}_{2,3} \simeq \mathfrak{sp}_4(\mathbb{R}),\ \mathfrak{so}_6(\mathbb{C}) \simeq \mathfrak{sl}_4(\mathbb{C}),\ \mathfrak{so}_6 \simeq \mathfrak{su}_4, \\ \mathfrak{so}_{1,5} \simeq \mathfrak{sl}_2(\mathbb{H}),\ \mathfrak{so}_{2,4} \simeq \mathfrak{su}_{2,2},\ \mathfrak{so}_{3,3} \simeq \mathfrak{sl}_4(\mathbb{R}),\ \mathfrak{u}_3(\mathbb{H}) \simeq \mathfrak{su}_{1,3},\ \mathfrak{u}_4(\mathbb{H}) \simeq \mathfrak{so}_{2,6}. \end{cases} \tag{18}$$

By definition, *classical complex Lie algebras* $\mathfrak{g}$ are the algebras $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{so}_n(\mathbb{C})$ and $\mathfrak{sp}_n(\mathbb{C})$. According to E. Cartan's classification (cf., e.g., [OV]), up to isomorphism, all their real forms $\mathfrak{g}_{\mathbb{R}}$ are listed in Table 1.

TABLE 1

$\mathfrak{g}$	$\mathfrak{g}_{\mathbb{R}}$	compact $\mathfrak{g}_{\mathbb{R}}$
$\mathfrak{sl}_n(\mathbb{C}),\ n\geqslant 2$	$\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{sl}_l(\mathbb{H}),\ n=2l$, $\mathfrak{su}_{n-q,q},\ q=0,1,\ldots,[n/2]$	$\mathfrak{su}_n := \mathfrak{su}_{n,0}$
$\mathfrak{so}_n(\mathbb{C}),\ n=3$ or $n\geqslant 5$	$\mathfrak{so}_{n-q,q},\ q=0,1,\ldots,[n/2]$, $\mathfrak{u}_l^*(\mathbb{H}),\ n=2l$	$\mathfrak{so}_n := \mathfrak{so}_{n,0}$
$\mathfrak{sp}_n(\mathbb{C}),\ n=2l\geqslant 2$	$\mathfrak{sp}_n(\mathbb{R})$, $\mathfrak{sp}_{l-q,q},\ q=0,1,\ldots,[l/2]$	$\mathfrak{sp}_l := \mathfrak{sp}_{l,0}$

There are the isomorphisms between some of the $\mathfrak{g}_{\mathbb{R}}$'s in Table 1 given by (18).

Now we restate the classification of nilpotent $\mathrm{Ad}(\mathfrak{g}_{\mathbb{R}})$-orbits in all real forms $\mathfrak{g}_{\mathbb{R}}$ of classical complex Lie algebras $\mathfrak{g}$ (cf. [BoCu], [CM], [SpSt], [M], [W]) in the form adapted to our goal of classifying compact nilpotent elements in $\mathfrak{g}_{\mathbb{R}}$.

If a subalgebra of $\mathfrak{g}_{\mathbb{R}}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, we call it an $\mathfrak{sl}_2(\mathbb{R})$-*subalgebra*. We use the following basic facts, cf. [CM], [KR], [M], [SpSt]:

(F1) For any nonzero nilpotent element $x\in\mathfrak{g}_{\mathbb{R}}$, there is an $\mathfrak{sl}_2(\mathbb{R})$-subalgebra of $\mathfrak{g}_{\mathbb{R}}$ containing x.

(F2) If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are the $\mathfrak{sl}_2(\mathbb{R})$-subalgebras of $\mathfrak{g}_{\mathbb{R}}$, and a nonzero nilpotent $\mathrm{Ad}(\mathfrak{g}_{\mathbb{R}})$-orbit intersects both $\mathfrak{a}_1$ and $\mathfrak{a}_2$, then $\mathfrak{a}_2 = g\cdot\mathfrak{a}_1$ for some $g\in\mathrm{Ad}(\mathfrak{g}_{\mathbb{R}})$.

(F3) There are exactly two nonzero nilpotent $\mathrm{Ad}(\mathfrak{sl}_2(\mathbb{R}))$-orbits in $\mathfrak{sl}_2(\mathbb{R})$. Scalar multiplication by -1 maps one of them to the other.

(F4) Any finite dimensional $\mathfrak{sl}_2(\mathbb{R})$-$D$-module is completely reducible. For any integer $d \in \mathbb{N}$, there is a unique, up to isomorphism, d-dimensional simple $\mathfrak{sl}_2(\mathbb{R})$-$D$-module S_d. The module S_1 is trivial. Let T_m be the direct sum of m copies of S_1, and $T_0 := 0$.

(F5) If $\mathfrak{a}$ is an $\mathfrak{sl}_2(\mathbb{R})$-subalgebra of $\mathfrak{g}_{\mathbb{R}}$ and $x \in \mathfrak{a}$ is a nonzero nilpotent element, then $\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{a})$ is the reductive Levi subalgebra of $\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(x)$.

(F6) If there is a nonzero $\mathfrak{sl}_2(\mathbb{R})$-invariant form on S_d of a given type (symmetric, skew-symmetric, Hermitian or skew-Hermitian), it is nondegenerate and unique up to proportionality. We fix such a form. Table 2 contains information about its existence and the notation for the fixed forms.

TABLE 2

D	d	symmetric	sgn	skew−symmetric	Hermitian	sgn	skew−Hermitian
$\mathbb{R}$	even			Δ_d^{ss}			
	odd	Δ_d^{s}	1				
$\mathbb{C}$	even			Δ_d^{ss}	Δ_d^{H}	0	$i\Delta_d^{H}$
	odd	Δ_d^{s}				1	
$\mathbb{H}$	even						Δ_d^{sH}
	odd				Δ_d^{H}	1	

Partitions, $\mathfrak{sl}_2(\mathbb{R})$-$D$-modules, and $\mathfrak{sl}_2(\mathbb{R})$-invariant forms

We call any vector

$$\boldsymbol{m} = (m_1, \ldots, m_p) \in \mathbb{N}^p, \quad \text{where } m_p \neq 0, \tag{19}$$

a *partition* of the number

$$|\boldsymbol{m}| := \textstyle\sum_d dm_d$$

(this is nontraditional usage of the term "partition" but it is convenient for our purposes). If $p = 1$, then $\boldsymbol{m}$ is called a *trivial* partition.

The *Young diagram* of $\boldsymbol{m}$ is a left-justified array $\mathrm{Y}(\boldsymbol{m})$ of empty boxes with p boxes in each of the first m_p rows, $p-1$ boxes in each of the next m_{p-1} rows, and so on. The partition

$$\check{\boldsymbol{m}} = \{\check{m}_1, \ldots, \check{m}_q\}$$

is called the *transpose partition* to $\boldsymbol{m}$ if $\mathrm{Y}(\check{\boldsymbol{m}})$ is the transpose of $\mathrm{Y}(\boldsymbol{m})$, i.e., the rows of $\mathrm{Y}(\check{\boldsymbol{m}})$ are the columns of $\mathrm{Y}(\boldsymbol{m})$ from left to right. We have $|\check{\boldsymbol{m}}| = |\boldsymbol{m}|$.

Let $\boldsymbol{m}$ be a nontrivial partition. It is called a *symmetric* (resp., *skew-symmetric*) *partition* of $|\boldsymbol{m}|$ if m_d in (19) is even for every even (resp., odd) d.

Let $\underline{\boldsymbol{m}}$ be a sequence obtained from $\boldsymbol{m}$ by replacing m_d in (19) for every d (resp., every odd d, every even d) with a pair $(p_d, q_d) \in \mathbb{N}^2$ such that $p_d + q_d = m_d$. Such $\underline{\boldsymbol{m}}$ is called a *fine* (resp., *fine Hermitian*, *fine skew-Hermitian*) *partition* of $|\boldsymbol{m}|$ *associated* with $\boldsymbol{m}$. If $\underline{\boldsymbol{m}}$ is fine or fine Hermitian,

$$\operatorname{sgn} \underline{\boldsymbol{m}} := \textstyle\sum_{d \text{ odd}} (p_d - q_d) \tag{20}$$

is called the *signature* of $\underline{\boldsymbol{m}}$. If $\underline{\boldsymbol{m}}$ is fine Hermitian (resp., fine skew-Hermitian) and $\boldsymbol{m}$ is symmetric (resp., skew-symmetric), then $\underline{\boldsymbol{m}}$ is called a *fine symmetric* (resp., *fine skew-symmetric*) *partition* of $|\boldsymbol{m}|$.

Any partition (19) defines the $|\boldsymbol{m}|$-dimensional $\mathfrak{sl}_2(\mathbb{R})$-$D$-module

$$V_{\boldsymbol{m}} := \bigoplus\nolimits_{d\geqslant 1}(T_{m_d}\otimes S_d) \tag{21}$$

(tensor product is taken over D, with respect to the canonical D-bimodule structure on D-vector spaces). By (F4), any nonzero finite dimensional $\mathfrak{sl}_2(\mathbb{R})$-$D$-module is isomorphic to $V_{\boldsymbol{m}}$ for a unique $\boldsymbol{m}$. It is trivial if and only if $\boldsymbol{m}$ is trivial.

Identifying every $\varphi \in \mathfrak{gl}_D(T_{m_d})$ with the transformation of $V_{\boldsymbol{m}}$ acting as $\varphi\otimes \mathrm{id}$ on the summand $T_{m_d}\otimes S_d$ and as 0 on the other summands in the right-hand side of (21), we identify $\oplus_{d\geqslant 1}\,\mathfrak{gl}_D(T_{m_d})$ with the subalgebra of $\mathfrak{gl}_D(V_{\boldsymbol{m}})$.

Two $\mathfrak{sl}_2(\mathbb{R})$-invariant forms on $V_{\boldsymbol{m}}$ are called $\mathfrak{sl}_2(\mathbb{R})$-*equivalent* if there is an $\mathfrak{sl}_2(\mathbb{R})$-equivariant isometry $V_{\boldsymbol{m}}\to V_{\boldsymbol{m}}$ with respect to them. To describe the $\mathfrak{sl}_2(\mathbb{R})$-equivalency classes of $\mathfrak{sl}_2(\mathbb{R})$-invariant forms on $V_{\boldsymbol{m}}$, we fix, for every positive integer r and pair $(p,q)\in \mathbb{N}^2$ with $p+q=r$, a nondegenerate form on T_r whose notation, type and signature (if applicable) are specified in Table 3.

TABLE 3

D	symmetric	sgn	skew−symmetric	Hermitian	sgn	skew−Hermitian
$\mathbb{R}$	$\Theta^{s}_{p,q}$	$p-q$	Θ^{ss}_{r}, r even			
$\mathbb{C}$	Θ^{s}_{r}		Θ^{ss}_{r}, r even	$\Theta^{H}_{p,q}$	$p-q$	
$\mathbb{H}$				$\Theta^{H}_{p,q}$	$p-q$	Θ^{sH}_{r}

According to the above discussion, any nondegenerate form on T_r is equivalent to a unique form from Table 3.

If Ψ is a nondegenerate $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_{\boldsymbol{m}}$, then $\Psi|_d$, its restriction to the summand $T_{m_d}\otimes S_d$ in (21), is a nondegenerate $\mathfrak{sl}_2(\mathbb{R})$-invariant form of the same type (i.e., symmetric, skew-symmetric, Hermitian or skew-Hermitian) as Ψ, and different summands in (21) are orthogonal with respect to Ψ. If Ψ' is another nondegenerate $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_{\boldsymbol{m}}$, then Ψ and Ψ' are $\mathfrak{sl}_2(\mathbb{R})$-equivalent if and only if $\Psi|_d$ and $\Psi'|_d$ are $\mathfrak{sl}_2(\mathbb{R})$-equivalent for every d. If, for every d, a nondegenerate $\mathfrak{sl}_2(\mathbb{R})$-invariant form Φ_d on $T_{m_d}\otimes S_d$ is fixed, and all Φ_d's are of the same type, then there is a nondegenerate $\mathfrak{sl}_2(\mathbb{R})$-invariant form Ψ on $V_{\boldsymbol{m}}$ such that $\Psi|_d=\Phi_d$ for all d. This reduces describing the $\mathfrak{sl}_2(\mathbb{R})$-equivalency classes of nondegenerate $\mathfrak{sl}_2(\mathbb{R})$-invariant forms on $V_{\boldsymbol{m}}$ to describing that on $T_{m_d}\otimes S_d$ for all d. For any positive integers r and d, Table 4 describes all, up to $\mathfrak{sl}_2(\mathbb{R})$-equivalency, nondegenerate $\mathfrak{sl}_2(\mathbb{R})$-invariant forms on $T_r\otimes S_d$ (in these tables, $p+q=r$).

TABLE 4

D	d	symmetric	sgn	skew−symmetric	Hermitian	sgn	skew−Hermitian
$\mathbb{R}$	even	$\Theta_r^{ss} \otimes \Delta_d^{ss}$	0	$\Theta_{p,q}^{s} \otimes \Delta_d^{ss}$			
	odd	$\Theta_{p,q}^{s} \otimes \Delta_d^{s}$	$p-q$	$\Theta_r^{ss} \otimes \Delta_d^{s}$			
$\mathbb{C}$	even	$\Theta_r^{ss} \otimes \Delta_d^{ss}$		$\Theta_r^{s} \otimes \Delta_d^{ss}$	$\Theta_{p,q}^{H} \otimes \Delta_d^{H}$	0	
	odd	$\Theta_r^{s} \otimes \Delta_d^{s}$		$\Theta_r^{ss} \otimes \Delta_d^{s}$		$p-q$	
$\mathbb{H}$					$\Theta_r^{sH} \otimes \Delta_d^{sH}$	0	$\Theta_{p,q}^{H} \otimes \Delta_d^{sH}$
					$\Theta_{p,q}^{H} \otimes \Delta_d^{H}$	$p-q$	$\Theta_r^{sH} \otimes \Delta_d^{H}$

Returning back to the n-dimensional vector space V over D, fix a form Φ (not necessarily nondegenerate) on V. Let $\boldsymbol{m}$ be a nontrivial partition of n and let

$$\alpha_{\boldsymbol{m}} : \mathfrak{sl}_2(\mathbb{R}) \hookrightarrow \mathfrak{gl}_D(V_{\boldsymbol{m}}) \tag{22}$$

be the injection defining the $\mathfrak{sl}_2(\mathbb{R})$-$D$-module structure on $V_{\boldsymbol{m}}$. Let Ψ be an $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_{\boldsymbol{m}}$. If Ψ and Φ are equivalent and $\iota : V_{\boldsymbol{m}} \to V$ is an isometry with respect to Ψ and Φ, then the image of the homomorphism

$$\mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{gl}_D(V), \quad \varphi \mapsto \iota \circ \alpha_{\boldsymbol{m}}(\varphi) \circ \iota^{-1},$$

is an $\mathfrak{sl}_2(\mathbb{R})$-subalgebra of $\mathfrak{gl}_D^{\Phi}(V)$. The above discussion shows that any $\mathfrak{sl}_2(\mathbb{R})$-subalgebra of $\mathfrak{gl}_D^{\Phi}(V)$ is obtained in this way from some pair $\{\boldsymbol{m}, \Psi\}$. The $\mathfrak{sl}_2(\mathbb{R})$-subalgebras of $\mathfrak{gl}_D^{\Phi}(V)$ obtained in this way from the pairs $\{\boldsymbol{m}, \Psi\}$, $\{\boldsymbol{m}', \Psi'\}$ are $\mathrm{GL}_D^{\Phi}(V)$-conjugate if and only if $\boldsymbol{m} = \boldsymbol{m}'$, and Ψ and Ψ' are $\mathfrak{sl}_2(\mathbb{R})$-equivalent. This yields a bijection between the union of $\mathfrak{sl}_2(\mathbb{R})$-equivalency classes of $\mathfrak{sl}_2(\mathbb{R})$-invariant forms equivalent to Φ on $V_{\boldsymbol{m}}$, where $\boldsymbol{m}$ ranges over all nontrivial partitions of n, and the set of all $\mathrm{GL}_D^{\Phi}(V)$-conjugacy classes of $\mathfrak{sl}_2(\mathbb{R})$-subalgebras in $\mathfrak{gl}_D^{\Phi}(V)$. *Denote this bijection by* $\star$. If Ψ is either zero or nondegenerate, considering $\star$ leads to the classification of nilpotent $\mathrm{Ad}(\mathfrak{gl}_D^{\Phi}(V))$-orbits in $\mathfrak{gl}_D^{\Phi}(V)$ described below. In each case, we give the formulas for the orbit dimensions (one obtains them using (8) and [CM], [M]); then (2), (8) yield the dimensions of the corresponding K-orbits in $\mathcal{N}(\mathfrak{p})$.

• *Nilpotent orbits in* $\mathfrak{sl}_D(V)$

Take $\Phi = 0$. Then $\mathrm{GL}_D^{\Phi}(V) = \mathrm{GL}_D(V)$ and $\mathfrak{gl}_D^{\Phi}(V) = \mathfrak{gl}_D(V)$. Any $\mathfrak{sl}_2(\mathbb{R})$-subalgebra of $\mathfrak{gl}_D(V)$ is contained in $\mathfrak{sl}_D(V)$. For any nontrivial partition $\boldsymbol{m}$ of n, consider the $\mathrm{GL}_D^{\Phi}(V)$-conjugacy class of $\mathfrak{sl}_2(\mathbb{R})$-subalgebras in $\mathfrak{sl}_D(V)$ corresponding under $\star$ to the $\mathfrak{sl}_2(\mathbb{R})$-equivalency class of zero form on $V_{\boldsymbol{m}}$. Then by (F1)–(F3), [KR], the subset of $\mathfrak{gl}_D^{\Phi}(V)$ that is the union of all $\mathfrak{sl}_2(\mathbb{R})$-subalgebras from this class contains a unique nonzero nilpotent $\mathrm{GL}_D(V)$-orbit $\mathcal{O}_{\boldsymbol{m}}$.

$\mathrm{Ad}(\mathfrak{sl}_D(V))$-orbits in $\mathcal{N}(\mathfrak{sl}_D(V))$:

> $\mathcal{O}_{\boldsymbol{m}}$ is an $\mathrm{Ad}(\mathfrak{sl}_D(V))$-orbit, except that $\mathcal{O}_{\boldsymbol{m}}$ is the union of two $\mathrm{Ad}(\mathfrak{sl}_D(V))$-orbits $\mathcal{O}_{\boldsymbol{m}}^1$ and $\mathcal{O}_{\boldsymbol{m}}^2$ (that are the connected components of $\mathcal{O}_{\boldsymbol{m}}$) if $D = \mathbb{R}$ and $m_d = 0$ for every odd d. Such $\mathrm{Ad}(\mathfrak{sl}_D(V))$-orbits, taken over all partitions $\boldsymbol{m}$ of n, are pairwise different and exhaust all nonzero $\mathrm{Ad}(\mathfrak{sl}_D(V))$-orbits in $\mathcal{N}(\mathfrak{sl}_D(V))$.

Orbit dimensions:

$$n^2 - \sum_d d^2 \check{m}_d = \begin{cases} \dim_{\mathbb{R}} \mathcal{O}_{\boldsymbol{m}} & \text{for } D = \mathbb{R}, \\ \dim_{\mathbb{C}} \mathcal{O}_{\boldsymbol{m}} & \text{for } D = \mathbb{C}, \\ \dim_{\mathbb{R}} \mathcal{O}_{\boldsymbol{m}}/4 & \text{for } D = \mathbb{H}. \end{cases}$$

• *Nilpotent orbits in $\mathfrak{gl}_D^\Phi(V)$ for $D = \mathbb{R}$ and nondegenerate symmetric Φ*

Let $\underline{\boldsymbol{m}}$ be a fine symmetric partition of n associated with (19). Let $\Psi_{\underline{\boldsymbol{m}}}$ be the $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_{\boldsymbol{m}}$ such that for all d,

$$\Psi_{\underline{\boldsymbol{m}}}|_d = \begin{cases} \Theta^s_{p_d,q_d} \otimes \Delta^s_d & \text{if } d \text{ is odd}, \\ \Theta^{ss}_{m_d} \otimes \Delta^{ss}_d & \text{if } d \text{ is even}. \end{cases} \tag{23}$$

Then $\Psi_{\underline{\boldsymbol{m}}}$ is equivalent to Φ if and only if

$$\operatorname{sgn} \underline{\boldsymbol{m}} = \operatorname{sgn} \Phi. \tag{24}$$

If (24) holds, consider the $\mathrm{GL}_D^\Phi(V)$-conjugacy class of $\mathfrak{sl}_2(\mathbb{R})$-subalgebras in $\mathfrak{gl}_D^\Phi(V)$ corresponding under $\star$ to the $\mathfrak{sl}_2(\mathbb{R})$-equivalency class of $\Psi_{\underline{\boldsymbol{m}}}$. Then by (F1)–(F3), [KR], the union of all $\mathfrak{sl}_2(\mathbb{R})$-subalgebras from this class contains a unique nilpotent $\mathrm{GL}_D^\Phi(V)$-orbit $\mathcal{O}_{\underline{\boldsymbol{m}}}$.

$\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits in $\mathcal{N}(\mathfrak{gl}_D^\Phi(V))$:

> If (24) holds, $\mathcal{O}_{\underline{\boldsymbol{m}}}$ is an $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbit, except the following cases. If $m_d = 0$ for all odd d, then $\mathcal{O}_{\underline{\boldsymbol{m}}}$ is the union of four $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits $\mathcal{O}^1_{\underline{\boldsymbol{m}}}$, $\mathcal{O}^2_{\underline{\boldsymbol{m}}}$, $\mathcal{O}^3_{\underline{\boldsymbol{m}}}$, $\mathcal{O}^4_{\underline{\boldsymbol{m}}}$ that are the connected components of $\mathcal{O}_{\underline{\boldsymbol{m}}}$. If there is an odd d such that $m_d \neq 0$ and
>
> $$\text{either } \begin{cases} p_d = 0 & \text{for all } d \equiv 1 \bmod 4, \\ q_d = 0 & \text{for all } d \equiv 3 \bmod 4 \end{cases} \quad \text{or } \begin{cases} p_d = 0 & \text{for all } d \equiv 3 \bmod 4, \\ q_d = 0 & \text{for all } d \equiv 1 \bmod 4, \end{cases}$$
>
> then $\mathcal{O}_{\underline{\boldsymbol{m}}}$ is the union of two $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits $\mathcal{O}^1_{\underline{\boldsymbol{m}}}$, $\mathcal{O}^2_{\underline{\boldsymbol{m}}}$ that are the connected components of $\mathcal{O}_{\underline{\boldsymbol{m}}}$. Such $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits, taken over all fine symmetric partitions $\underline{\boldsymbol{m}}$ of n satisfying (24), are pairwise different and exhaust all nonzero $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits in $\mathcal{N}(\mathfrak{gl}_D^\Phi(V))$.

Orbit dimensions:

$$\dim_{\mathbb{R}} \mathcal{O}_{\underline{\boldsymbol{m}}} = \big(n^2 - n - \textstyle\sum_d d^2 \check{m}_d + \sum_{d \text{ odd}} m_d\big)/2.$$

• *Nilpotent orbits in $\mathfrak{gl}_D^\Phi(V)$ for $D = \mathbb{C}$ and nondegenerate symmetric Φ*

Let $\boldsymbol{m}$ be a symmetric partition of n. Let $\Psi_{\boldsymbol{m}}$ be the $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_{\boldsymbol{m}}$ such that for all d,

$$\Psi_{\boldsymbol{m}}|_d = \begin{cases} \Theta^s_{m_d} \otimes \Delta^s_d & \text{if } d \text{ is odd}, \\ \Theta^{ss}_{m_d} \otimes \Delta^{ss}_d & \text{if } d \text{ is even}. \end{cases}$$

Then $\Psi_{\boldsymbol{m}}$ is equivalent to Φ. Consider the $\mathrm{GL}_D^\Phi(V)$-conjugacy class of $\mathfrak{sl}_2(\mathbb{R})$-subalgebras in $\mathfrak{gl}_D^\Phi(V)$ corresponding under $\star$ to the $\mathfrak{sl}_2(\mathbb{R})$-equivalency class of $\Psi_{\boldsymbol{m}}$. Then by (F1)–(F3), [KR], the union of all $\mathfrak{sl}_2(\mathbb{R})$-subalgebras from this class contains a unique nilpotent $\mathrm{GL}_D^\Phi(V)$-orbit $\mathcal{O}_{\boldsymbol{m}}$.

$\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits in $\mathcal{N}(\mathfrak{gl}_D^\Phi(V))$:

> $\mathcal{O}_{\boldsymbol{m}}$ is an $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbit, except that if $m_d = 0$ for all odd d, then $\mathcal{O}_{\boldsymbol{m}}$ is the union of two $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits $\mathcal{O}^1_{\boldsymbol{m}}$ and $\mathcal{O}^2_{\boldsymbol{m}}$ that are the connected components of $\mathcal{O}_{\boldsymbol{m}}$. Such $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits, taken over all symmetric partitions $\boldsymbol{m}$ of n, are pairwise different and exhaust all nonzero $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits in $\mathcal{N}(\mathfrak{gl}_D^\Phi(V))$.

Orbit dimensions:

$$\dim_{\mathbb{C}} \mathcal{O}_{\boldsymbol{m}} = \big(n^2 - n - \textstyle\sum_d d^2 \check{m}_d + \sum_{d \text{ odd}} m_d\big)/2.$$

• *Nilpotent orbits in $\mathfrak{gl}_D^\Phi(V)$ for $D = \mathbb{R}$ and nondegenerate skew-symmetric Φ*

Let $\underline{\boldsymbol{m}}$ be a fine skew-symmetric partition of n associated with (19). Let $\Psi_{\underline{\boldsymbol{m}}}$ be the $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_{\boldsymbol{m}}$ such that for all d,

$$\Psi_{\underline{\boldsymbol{m}}}|_d = \begin{cases} \Theta^s_{p_d,q_d} \otimes \Delta^{ss}_d & \text{if } d \text{ is even,}\\ \Theta^{ss}_{m_d} \otimes \Delta^s_d & \text{if } d \text{ is odd.}\end{cases} \tag{25}$$

Then $\Psi_{\underline{\boldsymbol{m}}}$ is equivalent to Φ. Consider the $\mathrm{GL}_D^\Phi(V)$-conjugacy class of $\mathfrak{sl}_2(\mathbb{R})$-subalgebras in $\mathfrak{gl}_D^\Phi(V)$ corresponding under $\star$ to the $\mathfrak{sl}_2(\mathbb{R})$-equivalency class of $\Psi_{\underline{\boldsymbol{m}}}$. Then by (F1)–(F3), [KR], the union of all $\mathfrak{sl}_2(\mathbb{R})$-subalgebras from this class contains a unique nilpotent $\mathrm{GL}_D^\Phi(V)$-orbit $\mathcal{O}_{\underline{\boldsymbol{m}}}$.

$\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits in $\mathcal{N}(\mathfrak{gl}_D^\Phi(V))$:

> $\mathcal{O}_{\underline{\boldsymbol{m}}}$ is an $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbit. Such $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits, taken over all fine skew-symmetric partitions $\underline{\boldsymbol{m}}$ of n, are pairwise different and exhaust all nonzero $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits in $\mathcal{N}(\mathfrak{gl}_D^\Phi(V))$.

Orbit dimensions:

$$\dim_{\mathbb{R}} \mathcal{O}_{\underline{\boldsymbol{m}}} = \big(n^2 + n - \textstyle\sum_d d^2 \check{m}_d - \sum_{d \text{ odd}} m_d\big)/2.$$

• *Nilpotent orbits in $\mathfrak{gl}_D^\Phi(V)$ for $D = \mathbb{C}$ and nondegenerate skew-symmetric Φ*

Let $\boldsymbol{m}$ be a skew-symmetric partition of n. Let $\Psi_{\boldsymbol{m}}$ be the $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_{\boldsymbol{m}}$ such that for all d,

$$\Psi_{\underline{\boldsymbol{m}}}|_d = \begin{cases} \Theta^s_{m_d} \otimes \Delta^{ss}_d & \text{if } d \text{ is even,}\\ \Theta^{ss}_{m_d} \otimes \Delta^s_d & \text{if } d \text{ is odd.}\end{cases}$$

Then $\Psi_{\underline{\boldsymbol{m}}}$ is equivalent to Φ. Consider the $\mathrm{GL}_D^\Phi(V)$-conjugacy class of $\mathfrak{sl}_2(\mathbb{R})$-subalgebras in $\mathfrak{gl}_D^\Phi(V)$ corresponding under $\star$ to the $\mathfrak{sl}_2(\mathbb{R})$-equivalency class of $\Psi_{\boldsymbol{m}}$. Then by (F1)–(F3), [KR], the union of all $\mathfrak{sl}_2(\mathbb{R})$-subalgebras from this class contains a unique nilpotent $\mathrm{GL}_D^\Phi(V)$-orbit $\mathcal{O}_{\boldsymbol{m}}$.

$\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits in $\mathcal{N}(\mathfrak{gl}_D^\Phi(V))$:

> $\mathcal{O}_{\boldsymbol{m}}$ is an $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbit. Such $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits, taken over all skew-symmetric partitions $\boldsymbol{m}$ of n, are pairwise different and exhaust all nonzero $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits in $\mathcal{N}(\mathfrak{gl}_D^\Phi(V))$.

Orbit dimensions:

$$\dim_{\mathbb{C}} \mathcal{O}_{\boldsymbol{m}} = \big(n^2 + n - \textstyle\sum_d d^2 \check{m}_d - \sum_{d \text{ odd}} m_d\big)/2.$$

• *Nilpotent orbits in* $\mathfrak{gl}_D^\Phi(V)$ *for* $D = \mathbb{C}$ *and nondegenerate Hermitian* Φ

Let $\underline{\boldsymbol{m}}$ be a fine partition of n associated with (19). Let $\Psi_{\underline{\boldsymbol{m}}}$ be the $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_{\boldsymbol{m}}$ such that for all d,

$$\Psi_{\underline{\boldsymbol{m}}}|_d = \Theta^H_{p_d,q_d} \otimes \Delta^H_d. \tag{26}$$

Then $\Psi_{\underline{\boldsymbol{m}}}$ is equivalent to Φ if and only if (24) holds. In the last case, consider the $\mathrm{GL}_D^\Phi(V)$-conjugacy class of $\mathfrak{sl}_2(\mathbb{R})$-subalgebras in $\mathfrak{gl}_D^\Phi(V)$ corresponding under $\star$ to the $\mathfrak{sl}_2(\mathbb{R})$-equivalency class of $\Psi_{\underline{\boldsymbol{m}}}$. Then by (F1)–(F3), [KR], the union of all $\mathfrak{sl}_2(\mathbb{R})$-subalgebras from this class contains a unique nilpotent $\mathrm{GL}_D^\Phi(V)$-orbit $\mathcal{O}_{\underline{\boldsymbol{m}}}$.

$\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits in $\mathcal{N}(\mathfrak{gl}_D^\Phi(V))$:

> If (24) holds, $\mathcal{O}_{\underline{\boldsymbol{m}}}$ is an $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbit. Such $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits, taken over all fine partitions $\underline{\boldsymbol{m}}$ of n satisfying (24), are pairwise different and exhaust all nonzero $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits in $\mathcal{N}(\mathfrak{gl}_D^\Phi(V))$.

Orbit dimensions:

$$\dim_{\mathbb{R}} \mathcal{O}_{\underline{\boldsymbol{m}}} = n^2 - \textstyle\sum_d d^2 \check{m}_d.$$

• *Nilpotent orbits in* $\mathfrak{gl}_D^\Phi(V)$ *for* $D = \mathbb{H}$ *and nondegenerate Hermitian* Φ

Let $\underline{\boldsymbol{m}}$ be a fine Hermitian partition of n associated with (19). Let $\Psi_{\underline{\boldsymbol{m}}}$ be the $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_{\boldsymbol{m}}$ such that for all d,

$$\Psi_{\underline{\boldsymbol{m}}}|_d = \begin{cases} \Theta^{sH}_{m_d} \otimes \Delta^{sH}_d & \text{if } d \text{ is even,} \\ \Theta^{H}_{p_d,q_d} \otimes \Delta^{H}_d & \text{if } d \text{ is odd.} \end{cases} \tag{27}$$

Then $\Psi_{\underline{\boldsymbol{m}}}$ is equivalent to Φ if and only if (24) holds. In the last case, consider the $\mathrm{GL}_D^\Phi(V)$-conjugacy class of $\mathfrak{sl}_2(\mathbb{R})$-subalgebras in $\mathfrak{gl}_D^\Phi(V)$ corresponding under $\star$ to the $\mathfrak{sl}_2(\mathbb{R})$-equivalency class of $\Psi_{\underline{\boldsymbol{m}}}$. Then by (F1)–(F3), [KR], the union of all $\mathfrak{sl}_2(\mathbb{R})$-subalgebras from this class contains a unique nilpotent $\mathrm{GL}_D^\Phi(V)$-orbit $\mathcal{O}_{\underline{\boldsymbol{m}}}$.

$\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits in $\mathcal{N}(\mathfrak{gl}_D^\Phi(V))$:

> If (24) holds, $\mathcal{O}_{\underline{\boldsymbol{m}}}$ is an $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbit. Such $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits, taken over all fine Hermitian partitions $\underline{\boldsymbol{m}}$ of n satisfying (24), are pairwise different and exhaust all nonzero $\mathrm{Ad}(\mathfrak{gl}_D^\Phi(V))$-orbits in $\mathcal{N}(\mathfrak{gl}_D^\Phi(V))$.

Orbit dimensions:

$$\dim_{\mathbb{R}} \mathcal{O}_{\underline{\boldsymbol{m}}} = 2n^2 + n - 2\textstyle\sum_d d^2 \check{m}_d - \sum_{d \text{ odd}} m_d.$$

• *Nilpotent orbits in $\mathfrak{gl}_D^{\Phi}(V)$ for $D=\mathbb{H}$ and skew-Hermitian Φ*

Let $\underline{\boldsymbol{m}}$ be a fine skew-Hermitian partition of n associated with (19). Let $\Psi_{\underline{\boldsymbol{m}}}$ be the $\mathfrak{sl}_2(\mathbb{R})$-invariant form on $V_{\boldsymbol{m}}$ such that for all d,

$$\Psi_{\underline{\boldsymbol{m}}}|_d = \begin{cases} \Theta^{H}_{p_d,q_d}\otimes\Delta_d^{sH} & \text{if } d \text{ is even},\\ \Theta^{sH}_{m_d}\otimes\Delta_d^{H} & \text{if } d \text{ is odd}.\end{cases} \tag{28}$$

Then $\Psi_{\underline{\boldsymbol{m}}}$ is equivalent to Φ. Consider the $\mathrm{GL}_D^{\Phi}(V)$-conjugacy class of $\mathfrak{sl}_2(\mathbb{R})$-subalgebras in $\mathfrak{gl}_D^{\Phi}(V)$ corresponding under $\star$ to the $\mathfrak{sl}_2(\mathbb{R})$-equivalency class of $\Psi_{\underline{\boldsymbol{m}}}$. Then by (F1)–(F3), [KR], the union of all $\mathfrak{sl}_2(\mathbb{R})$-subalgebras from this class contains a unique nilpotent $\mathrm{GL}_D^{\Phi}(V)$-orbit $\mathcal{O}_{\underline{\boldsymbol{m}}}$.

$\mathrm{Ad}(\mathfrak{gl}_D^{\Phi}(V))$-orbits in $\mathcal{N}(\mathfrak{gl}_D^{\Phi}(V))$:

> $\mathcal{O}_{\underline{\boldsymbol{m}}}$ is an $\mathrm{Ad}(\mathfrak{gl}_D^{\Phi}(V))$-orbit. Such $\mathrm{Ad}(\mathfrak{gl}_D^{\Phi}(V))$-orbits, taken over all fine skew-Hermitian partitions $\underline{\boldsymbol{m}}$ of n, are pairwise different and exhaust all $\mathrm{Ad}(\mathfrak{gl}_D^{\Phi}(V))$-orbits in $\mathcal{N}(\mathfrak{gl}_D^{\Phi}(V))$.

Orbit dimensions:

$$\dim_{\mathbb{R}}\mathcal{O}_{\underline{\boldsymbol{m}}} = 2n^2-n-2\textstyle\sum_d d^2\check{m}_d+\sum_{d \text{ odd}}\, m_d.$$

Now we are ready to classify compact nilpotent elements in the real forms of complex classical Lie algebras. Recall that $n=\dim_D V$.

• *Compact nilpotent elements in $\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{sl}_n(\mathbb{H})$*

Theorem 8. *Let $\mathcal{O}$ be the $\mathrm{Ad}(\mathfrak{sl}_D(V))$-orbit of a nonzero element $x\in\mathcal{N}(\mathfrak{sl}_D(V))$ where D is $\mathbb{R}$ or $\mathbb{H}$. Let $\mathfrak{a}$ be an $\mathfrak{sl}_2(\mathbb{R})$-subalgebra of $\mathfrak{sl}_D(V)$ containing x. The following properties are equivalent:*

(i) *x is compact.*
(ii) *$\mathfrak{a}$-module V is simple.*
(iii) *If $D=\mathbb{R}$, then $\mathcal{O}=\mathcal{O}_{(0,\dots,0,1)}$ for odd n, and $\mathcal{O}^1_{(0,\dots,0,1)}$ or $\mathcal{O}^2_{(0,\dots,0,1)}$ for even n. If $D=\mathbb{H}$, then $\mathcal{O}=\mathcal{O}_{(0,\dots,0,1)}$.*

Proof. According to the previous discussion, we may, and will, assume that $V=V_{\boldsymbol{m}}$ and $\mathfrak{a}=\alpha_{\boldsymbol{m}}(\mathfrak{sl}_2(\mathbb{R}))$ (see (22)) for some nontrivial partition $\boldsymbol{m}$ of n. The Double Centralizer Theorem implies that

$$\mathfrak{z}_{\mathfrak{gl}_D(V)}(\mathfrak{a}) = \textstyle\bigoplus_{d\geqslant 1}\ \mathfrak{gl}_D(T_{m_d}). \tag{29}$$

Since $\mathfrak{gl}_D(T_{m_d})$ (resp., $\mathfrak{sl}_D(T_{m_d})$) is compact if and only if $m_d=0$ (resp., 0 or 1), the claim follows from (29) and (F5). □

• *Compact nilpotent elements in $\mathfrak{so}_{n-q,q}$*

Theorem 9. *Let $\mathcal{O}$ be the $\mathrm{Ad}(\mathfrak{gl}_{\mathbb{R}}^{\Phi}(V))$-orbit of a nonzero element $x\in\mathcal{N}(\mathfrak{gl}_{\mathbb{R}}^{\Phi}(V))$ for a nondegenerate symmetric form Φ. The following properties are equivalent:*

(i) *x is compact.*

(ii) $\mathcal{O} = \mathcal{O}^1_{\underline{\boldsymbol{m}}}$ *or* $\mathcal{O}^2_{\underline{\boldsymbol{m}}}$, *where* $\underline{\boldsymbol{m}}$ *is a fine symmetric partition of* n *such that* (24) *holds,* $p_d q_d = 0$ *for all odd* d, *and* $m_d = 0$ *for all even* d.

Proof. We may, and will, assume that $V = V_{\boldsymbol{m}}$, Φ is $\Psi_{\underline{\boldsymbol{m}}}$ defined by (23), and $x \in \mathfrak{a} := \alpha_{\boldsymbol{m}}(\mathfrak{sl}_2(\mathbb{R}))$ for some fine symmetric partition $\underline{\boldsymbol{m}}$ of n such that (24) holds. Then (29) holds, whence by (23)

$$(30)\qquad \mathfrak{z}_{\mathfrak{gl}^{\Phi}_{\mathbb{R}}(V)}(\mathfrak{a}) = \bigoplus\nolimits_{d\geqslant 1} \mathfrak{gl}^{\Theta_d}_{\mathbb{R}}(T_{m_d}) \ \text{ where } \ \Theta_d = \begin{cases} \Theta^s_{p_d,q_d} & \text{if } d \text{ is odd,} \\ \Theta^{ss}_{m_d} & \text{if } d \text{ is even.} \end{cases}$$

Since $\mathfrak{gl}^{\Theta_d}_{\mathbb{R}}(T_{m_d})$ for Θ_d given by (30) is compact if and only if $p_d q_d = 0$ for odd d and $m_d = 0$ for even d (see Table 1 and (18)), the claim follows from (30) and (F5). $\square$

- *Compact nilpotent elements in* $\mathfrak{sp}_n(\mathbb{R})$

Theorem 10. *Let* $\mathcal{O}$ *be the* $\mathrm{Ad}(\mathfrak{gl}^{\Phi}_{\mathbb{R}}(V))$*-orbit of a nonzero element* $x \in \mathcal{N}(\mathfrak{gl}^{\Phi}_{\mathbb{R}}(V))$ *for a nondegenerate skew-symmetric form* Φ. *The following properties are equivalent:*

(i) x *is compact.*

(ii) $\mathcal{O} = \mathcal{O}_{\underline{\boldsymbol{m}}}$ *where* $\underline{\boldsymbol{m}}$ *is a fine skew-symmetric partition of* n *such that* $p_d q_d = 0$ *for all even* d, *and* $m_d = 0$ *for all odd* d.

Proof. The arguments are similar to that in the proof of Theorem 9 with (23) replaced with (25). $\square$

- *Compact nilpotent elements in* $\mathfrak{su}_{n-q,q}$

Theorem 11. *Let* $\mathcal{O}$ *be the* $\mathrm{Ad}(\mathfrak{gl}^{\Phi}_{\mathbb{C}}(V))$*-orbit of a nonzero element* $x \in \mathcal{N}(\mathfrak{gl}^{\Phi}_{\mathbb{C}}(V))$ *for a nondegenerate Hermitian form* Φ. *The following properties are equivalent:*

(i) x *is compact.*

(ii) $\mathcal{O} = \mathcal{O}_{\underline{\boldsymbol{m}}}$ *where* $\underline{\boldsymbol{m}}$ *is a fine partition of* n *such that* (24) *holds and* $p_d q_d = 0$ *for all* d.

Proof. By Table 1, the algebra $\mathfrak{gl}^{\Psi}_{\mathbb{C}}(W)$ for $\Psi = \Theta^H_{p_d,q_d}$ and $W = T_{m_d}$ is compact if and only if $p_d q_d = 0$. Now one completes the proof using the arguments similar to that in the proof of Theorem 9 with (23) replaced with (26). $\square$

- *Compact nilpotent elements in* $\mathfrak{sp}_{n/2-q,q}$

Theorem 12. *Let* $\mathcal{O}$ *be the* $\mathrm{Ad}(\mathfrak{gl}^{\Phi}_{\mathbb{H}}(V))$*-orbit of a nonzero element* $x \in \mathcal{N}(\mathfrak{gl}^{\Phi}_{\mathbb{H}}(V))$ *for a nondegenerate Hermitian form* Φ. *The following properties are equivalent:*

(i) x *is compact.*

(ii) $\mathcal{O} = \mathcal{O}_{\underline{\boldsymbol{m}}}$ *where* $\underline{\boldsymbol{m}}$ *is a fine Hermitian partition of* n *such that* (24) *holds,* $p_d q_d = 0$ *for all odd* d, *and* $m_d = 0$ *or* 1 *for all even* d.

Proof. It follows from Table 1 and (18) that the algebra $\mathfrak{gl}^{\Psi}_{\mathbb{H}}(W)$ for $\Psi = \Theta^H_{p_d,q_d}$ (resp., $\Theta^{sH}_{m_d}$) and $W = T_{m_d}$ is compact if and only if $p_d q_d = 0$ (resp., $m_d = 0$ or 1). Now one completes the proof using the arguments similar to that in the proof of Theorem 9 with (23) replaced with (27). $\square$

- *Compact nilpotent elements in* $\mathfrak{u}_n^*(\mathbb{H})$

Theorem 13. *Let* $\mathcal{O}$ *be the* $\mathrm{Ad}(\mathfrak{gl}^\Phi_{\mathbb{H}}(V))$*-orbit of a nonzero element* $x \in \mathcal{N}(\mathfrak{gl}^\Phi_{\mathbb{H}}(V))$ *for a nondegenerate skew-Hermitian form* Φ*. The following properties are equivalent:*

(i) x *is compact.*

(ii) $\mathcal{O} = \mathcal{O}_{\underline{\boldsymbol{m}}}$ *where* $\underline{\boldsymbol{m}}$ *is a fine skew-Hermitian partition of* n *such that* $p_d q_d = 0$ *for all even* d*, and* $m_d = 0$ *or* 1 *for all odd* d*.*

Proof. The arguments are similar to that in the proof of Theorem 12 with (27) replaced with (28). □

5. Classification for exceptional simple $\mathfrak{g}$

In this section, we assume that $\mathfrak{g}$ is an exceptional simple complex Lie algebra and $\mathfrak{g}_{\mathbb{R}}$ is its noncompact real form. According to E. Cartan's classification (cf., e.g., [OV]), up to isomorphism there are, in total, twelve possibilities listed in Table 5. Its second column gives both types of E. Cartan's notation for $\mathfrak{g}_{\mathbb{R}}$; the notation $\mathtt{X}_{s(t)}$ means that $\mathtt{X}_s$ is the type of $\mathfrak{g}$ and t is the signature of the Killing form of $\mathfrak{g}_{\mathbb{R}}$, i.e., $t = \dim_{\mathbb{C}} \mathfrak{p} - \dim_{\mathbb{C}} \mathfrak{k}$.

TABLE 5

$\mathfrak{g}$	$\mathfrak{g}_{\mathbb{R}}$	type of $\mathfrak{k}_{\mathbb{R}}$	$\dim_{\mathbb{C}} \mathfrak{k}$	$\dim_{\mathbb{C}} \mathfrak{p}$
E_6	$\mathtt{EI}=\mathrm{E}_{6(6)}$	$\mathfrak{sp}_4$	36	42
E_6	$\mathtt{EII}=\mathrm{E}_{6(2)}$	$\mathfrak{su}_2 \oplus \mathfrak{su}_6$	38	40
E_6	$\mathtt{EIII}=\mathrm{E}_{6(-14)}$	$\mathfrak{so}_{10} \oplus \mathbb{R}$	46	32
E_6	$\mathtt{EIV}=\mathrm{E}_{6(-26)}$	$\mathrm{F}_{4(-52)}$	52	26
E_7	$\mathtt{EV}=\mathrm{E}_{7(7)}$	$\mathfrak{su}_8$	63	70
E_7	$\mathtt{EVI}=\mathrm{E}_{7(-5)}$	$\mathfrak{su}_2 \oplus \mathfrak{so}_{12}$	69	64
E_7	$\mathtt{EVII}=\mathrm{E}_{7(-25)}$	$\mathrm{E}_{6(-78)} \oplus \mathbb{R}$	79	54
E_8	$\mathtt{EVIII}=\mathrm{E}_{8(8)}$	$\mathfrak{so}_{16}$	120	128
E_8	$\mathtt{EIX}=\mathrm{E}_{8(-24)}$	$\mathfrak{su}_2 \oplus \mathrm{E}_{7(-133)}$	136	112
F_4	$\mathtt{FI}=\mathrm{F}_{4(4)}$	$\mathfrak{su}_2 \oplus \mathfrak{sp}_3$	24	28
F_4	$\mathtt{FII}=\mathrm{F}_{4(-20)}$	$\mathfrak{so}_9$	36	16
G_2	$\mathtt{GI}=\mathrm{G}_{2(2)}$	$\mathfrak{so}_3 \oplus \mathfrak{so}_3$	6	8

Recall the classical approach to classifying nilpotent orbits by means of their characteristics and weighted Dynkin diagrams, cf. [CM], [M]. Fix a θ-stable Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ such that $\mathfrak{t} \cap \mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{k}$. Denote

$$l := \dim_{\mathbb{C}} \mathfrak{t}, \quad s := \dim_{\mathbb{C}}(\mathfrak{t} \cap \mathfrak{k}).$$

Let $\Delta_{(\mathfrak{g},\mathfrak{t})}$ and $\Delta_{(\mathfrak{k},\mathfrak{t}\cap\mathfrak{k})}$ be the root systems of resp. $(\mathfrak{g}, \mathfrak{t})$ and $(\mathfrak{k}, \mathfrak{t} \cap \mathfrak{k})$. Given a nonzero element $e \in \mathcal{N}(\mathfrak{g})$ (resp., $e \in \mathcal{N}(\mathfrak{p})$), there are the elements $h, f \in \mathfrak{g}$ (resp., $h \in \mathfrak{k}$, $f \in \mathfrak{p}$)

such that $\{e, h, f\}$ is an $\mathfrak{sl}_2$-triple. The intersection of $\mathfrak{t}$ (resp., $\mathfrak{t}\cap\mathfrak{k}$) with the orbit $\mathcal{O} = G\cdot e$ (resp., $K \cdot e$) contains a unique element h_0 lying in a fixed Weyl chamber of $\Delta_{(\mathfrak{g},\mathfrak{t})}$ (resp., $\Delta_{(\mathfrak{k},\mathfrak{t}\cap\mathfrak{k})}$). The mapping $\mathcal{O} \mapsto h_0$ is a well defined injection of the set of all nonzero G-orbits in $\mathcal{N}(\mathfrak{g})$ (resp., K-orbits in $\mathcal{N}(\mathfrak{p})$) into $\mathfrak{t}$ (resp., $\mathfrak{t}\cap\mathfrak{k}$). The image h_0 of the orbit $\mathcal{O}$ under this injection is called the *characteristic* of $\mathcal{O}$ (and e). Thus the G-orbits in $\mathcal{N}(\mathfrak{g})$ (resp., K-orbits in $\mathcal{N}(\mathfrak{p})$) are defined by their characteristics. In turn, the characteristics are defined by the numerical data, namely, the system of values $\{\beta_j(h_0)\}$ where $\{\beta_j\}$ is a fixed basis of $\mathfrak{t}^*$ (resp., $(\mathfrak{t}\cap\mathfrak{k})^*$). In practice, $\{\beta_j\}$ is always chosen to be a base of some root system. Then assigning the integer $\beta_j(h_0)$ to each node β_j of the Dynkin diagram of this base yields the *weighted Dynkin diagram* of the orbit $\mathcal{O}$, denoted by $\operatorname{Dyn}\mathcal{O}$. It uniquely defines $\mathcal{O}$. This choice of the base $\{\beta_j\}$ is the following.

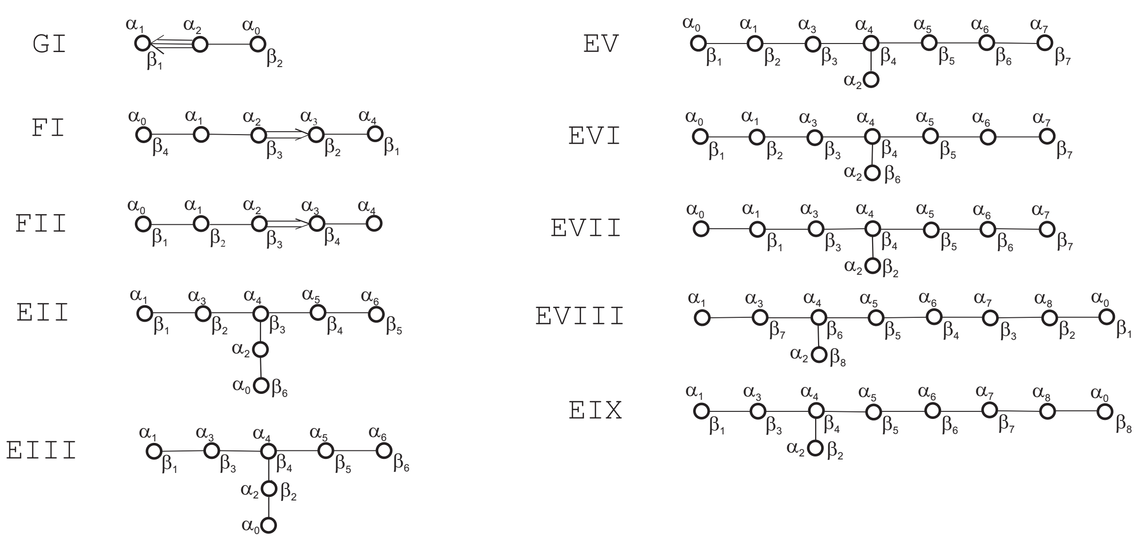
For $\mathfrak{t}^*$, since $\mathfrak{g}$ is semisimple, it is natural to take $\{\beta_j\}$ to be the base $\alpha_1, \dots, \alpha_l$ of $\Delta_{(\mathfrak{g},\mathfrak{t})}$ defining the fixed Weyl chamber that is used in the definition of characteristics. Below the Dynkin diagrams of G-orbits in $\mathcal{N}(\mathfrak{g})$ are considered with respect to this base $\alpha_1, \dots, \alpha_l$.

To describe $\{\beta_j\}$ for $(\mathfrak{t}\cap\mathfrak{k})^*$, denote by α_0 the lowest root of $\Delta_{(\mathfrak{g},\mathfrak{t})}$. The extended Dynkin diagrams of $\Delta_{(\mathfrak{g},\mathfrak{t})}$ indicating the location of the α_i's are given in Table 6.

The real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$ is of inner type (i.e., $s = l$, $\mathfrak{t} \subset \mathfrak{k}$, and thereby $\Delta_{(\mathfrak{k},\mathfrak{t}\cap\mathfrak{k})} \subset \Delta_{(\mathfrak{g},\mathfrak{t})}$) if and only if $\mathfrak{g}_{\mathbb{R}} \neq \mathtt{E}_{6(6)}$, $\mathtt{E}_{6(-26)}$. If $\mathfrak{g}_{\mathbb{R}}$ is of inner type, then $\mathfrak{k}$ is semisimple if and only if $\mathfrak{g}_{\mathbb{R}} \neq \mathtt{E}_{6(-14)}, \mathtt{E}_{7(-25)}$; in the last two cases the center of $\mathfrak{k}$ is one-dimensional. If $\mathfrak{g}_{\mathbb{R}}$ is of inner type, then the set $\{\alpha_0, \dots, \alpha_l\}$ contains a unique base $\beta_1, \dots, \beta_r$ of $\Delta_{(\mathfrak{k},\mathfrak{t}\cap\mathfrak{k})}$. Moreover, α_0 is one of the β_i's $\iff$ $\mathfrak{k}$ is semisimple $\iff$ $r = l$. If $\mathfrak{g}_{\mathbb{R}}$ is of inner type and $\mathfrak{k}$ is not semisimple, then $r = l - 1$ and $\beta_j \in \{\alpha_1, \dots, \alpha_l\}$ for all $j = 1, \dots, r$; in this case we set $\beta_l := \alpha_l$.

Thus in all cases where $\mathfrak{g}_{\mathbb{R}}$ is of inner type, we have a basis $\{\beta_j\}$ of $(\mathfrak{t}\cap\mathfrak{k})^*$ that is the system of simple roots of some root system. The location of the β_j's is given in Table 6.

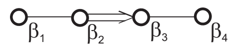
TABLE 6



If $\mathfrak{g}_{\mathbb{R}}$ is of outer type, then $\mathfrak{k}$ is semisimple, so we may, and will, take $\{\beta_j\}$ to be a base of $\Delta_{(\mathfrak{k},\mathfrak{t}\cap\mathfrak{k})}$. We take the following base. If $\mathfrak{g}_{\mathbb{R}} = \mathtt{E}_{6(-26)}$, then

$$\beta_4 = \alpha_1|_{\mathfrak{t}\cap\mathfrak{k}} = \alpha_6|_{\mathfrak{t}\cap\mathfrak{k}}, \quad \beta_3 = \alpha_3|_{\mathfrak{t}\cap\mathfrak{k}} = \alpha_5|_{\mathfrak{t}\cap\mathfrak{k}}, \quad \beta_2 = \alpha_4|_{\mathfrak{t}\cap\mathfrak{k}}, \quad \beta_1 = \alpha_2|_{\mathfrak{t}\cap\mathfrak{k}},$$

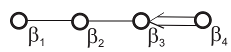
with the Dynkin diagram



and if $\mathfrak{g}_{\mathbb{R}} = \mathtt{E}_{6(6)}$, then

$$\beta_1 = -2\beta_2 - 3\beta_3 - 2\beta_4 - \alpha_2|_{\mathfrak{t}\cap\mathfrak{k}},$$
$$\beta_2 = \alpha_1|_{\mathfrak{t}\cap\mathfrak{k}} = \alpha_6|_{\mathfrak{t}\cap\mathfrak{k}}, \quad \beta_3 = \alpha_3|_{\mathfrak{t}\cap\mathfrak{k}} = \alpha_5|_{\mathfrak{t}\cap\mathfrak{k}}, \quad \beta_4 = \alpha_4|_{\mathfrak{t}\cap\mathfrak{k}}$$

with the Dynkin diagram



Below the weighted Dynkin diagrams of K-orbits in $\mathcal{N}(\mathfrak{g})$ are considered with respect to the described base $\{\beta_j\}$.

In [D3], [D4] one finds:

(D1) the weighted Dynkin diagrams of all nonzero orbits $G \cdot x$ and $K \cdot x$ for $x \in \mathcal{N}(\mathfrak{p})$,
(D2) the type and dimension of the reductive Levi factor of $\mathfrak{z}_{\mathfrak{k}}(x)$ and $\dim_{\mathbb{C}} \mathfrak{z}_{\mathfrak{k}}(x)$,
(D3) the type of the reductive Levi factor of $\mathfrak{z}_{\mathfrak{g}_{\mathbb{R}}}(y)$ for an element y of the $\mathrm{Ad}(\mathfrak{g}_{\mathbb{R}})$-orbit in $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})$ corresponding to $K \cdot x$ via the Kostant–Sekiguchi bijection.

Given the weighted Dynkin diagram of $G \cdot x$, one finds the type of reductive Levi factor of $\mathfrak{z}_{\mathfrak{g}}(x)$ in [El], [Ca]. So using (D1), one can apply Theorem 4 for explicit classifying (-1)-distinguished K-orbits in $\mathcal{N}(\mathfrak{p})$. On the other hand, given (D3), Theorem 5 can be applied for this purpose as well. So following either of these ways, one obtains, for every exceptional simple $\mathfrak{g}$, the explicit classification of (-1)-distinguished K-orbits in $\mathcal{N}(\mathfrak{p})$ in terms of their weighted Dynkin diagrams. The final result is the following.

Theorem 14. *For all exceptional simple Lie algebras $\mathfrak{g}$ and all conjugacy classes of elements $\theta \in \mathrm{Aut}\,\mathfrak{g}$ of order 2, all (-1)-distinguished K-orbits in $\mathcal{N}(\mathfrak{p})$ are listed in Tables 7–18 at the end of this paper.*

Tables 7–18 contain further information as summarized below.

(T1) The conjugacy class of θ is defined by specifying the type of noncompact real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$ canonically corresponding to this class.
(T2) Column 2 gives the weights $\{\beta_j(h)\}$ (listed in the order of increasing of j) of the weighted Dynkin diagram $\mathrm{Dyn}\,K \cdot x$ of the (-1)-distinguished K-orbit $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$, where h is the characteristic of $K \cdot x$.

(T3) Column 3 gives the weights $\{\alpha_j(H)\}$ (listed in the order of increasing of j) of the weighted Dynkin diagram $\operatorname{Dyn} G\cdot x$ of the G-orbit $G\cdot x$ in $\mathcal{N}(\mathfrak{g})$, where H is the characteristic of $G\cdot x$.

(T4) Column 4 gives $\dim_{\mathbb{C}} K\cdot x$, and hence, by (2), also $\dim_{\mathbb{C}} G\cdot x$.

(T5) Column 5 gives the number of K-orbits (not necessarily (-1)-distinguished) in $G\cdot x\cap\mathfrak{p}$. One of them is $K\cdot x$. Since this number is equal to the number of K-orbits $K\cdot x'$ in $\mathcal{N}(\mathfrak{p})$ such that $\operatorname{Dyn} G\cdot x' = \operatorname{Dyn} G\cdot x$, one finds it using the tables in [D3], [D4].

(T6) Columns 6 gives the type of the reductive Levi factor of $\mathfrak{z}_{\mathfrak{k}}(x)$. By $\mathtt{T}_m$ is denoted the Lie algebra of an m-dimensional torus.

(T7) Column 7 gives the complex dimension of the unipotent radical of $\mathfrak{z}_{\mathfrak{k}}(x)$.

Remark 1. For $\mathtt{E}_{6(6)}$ and $\mathtt{E}_{6(-26)}$, our numeration of $\{\alpha_j\}$ is the same as in [D3]; it differs from that in [D4]. For $\mathtt{E}_{6(-26)}$, our numeration of $\{\beta_j\}$ differs from that in [D3].

Remark 2. One finds in [D5], [D6], [D7] the explicit classification of all Cayley triples in $\mathfrak{g}$. This yields the explicit representatives of all K-orbits in $\mathcal{N}(\mathfrak{p})$ and $\operatorname{Ad}(\mathfrak{g}_{\mathbb{R}})$-orbits in $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})$.

6. Geometric properties

Geometric properties of the varieties $\mathbf{P}(\overline{\mathcal{O}})$ where $\mathcal{O}$ is a nilpotent K-orbit in $\mathcal{N}(\mathfrak{p})$ have been studies by several authors. Thereby their results provide some information on the geometry of the projective self-dual varieties that we associated with symmetric spaces. However these results are less complete than in the adjoint case (where a rather detailed information about singular loci and normality of the projective self-dual varieties $\mathbf{P}(\overline{\mathcal{O}})$ is available, see [P2]). Some phenomena valid for the projective self-dual $\mathbf{P}(\overline{\mathcal{O}})$'s in the adjoint case fail in general. Below we briefly summarize some facts about the geometry of the projective self-dual varieties that we associated with symmetric spaces.

Intersections of G-orbits in $\mathbf{P}(\mathcal{N}(\mathfrak{g}))$ with the linear subspace $\mathbf{P}(\mathfrak{p})$

Let $\mathcal{O}$ be a nonzero G-orbit in $\mathcal{N}(\mathfrak{g})$, and $X=\mathbf{P}(\overline{\mathcal{O}})$. If $X\cap\mathbf{P}(\mathfrak{p})\neq\varnothing$, then, by (2), all irreducible components $Y_1,\dots Y_s$ of the variety $X\cap\mathbf{P}(\mathfrak{p})$ have dimension $\frac{1}{2}\dim X$, and every Y_j is the closure of a K-orbit in $\mathbf{P}(\mathcal{N}(\mathfrak{p}))$. Theorems 2, 3 and Definition 1 imply that

$$X \text{ is self-dual} \quad\Longrightarrow\quad \text{all } Y_1,\dots,Y_s \text{ are self-dual.}$$

There are many examples showing that the converse in not true. For instance, one deduces from Table 8 that $\mathfrak{g}=\mathtt{E}_6$, $\mathfrak{g}_{\mathbb{R}}=\mathtt{E}_{6(2)}$ and $\operatorname{Dyn}\mathcal{O}=111011, 121011$ or 220202 are such cases. There are also many examples where some of the Y_j's are self-dual and some are not: for instance, this is so if $\mathfrak{g}=\mathtt{E}_6$, $\mathfrak{g}_{\mathbb{R}}=\mathtt{E}_{6(2)}$ and $\operatorname{Dyn}\mathcal{O}=020000, 001010, 000200, 020200$ or 220002. Finally, there are many instances where all Y_j are not self-dual (to obtain them, e.g., compare Tables 7–18 with tables in [D3], [D4]).

Affine (-1)-distinguished orbits

Recall that by Matsushima's criterion, an orbit of a reductive algebraic group acting on an affine algebraic variety is affine if and only if the stabilizer of a point of this orbit is reductive, cf., e.g. [PV, Theorem 4.17]. Therefore distinguished G-orbits in $\mathcal{N}(\mathfrak{g})$ are never

affine. However in general there exist affine (-1)-distinguished K-orbits in $\mathcal{N}(\mathfrak{p})$. For instance, if $\mathfrak{g}$ is exceptional simple, we immediately obtain their classification from Tables 7–18: these are precisely the orbits for which the number in the last column is 0.

If $K \cdot x$ is affine, then each irreducible component of the boundary $\mathbf{P}(\overline{K \cdot x}) \setminus \mathbf{P}(K \cdot x)$ has codimension 1 in $\mathbf{P}(\overline{K \cdot x})$, cf. [P1, Lemma 3]. Therefore if a point of the open K-orbit of such an irreducible component lies in the singular locus of $\mathbf{P}(\overline{K \cdot x})$, then $\mathbf{P}(\overline{K \cdot x})$ is not normal.

Orbit closure ordering and orbit decomposition of the orbit boundary $\mathbf{P}(\overline{K \cdot x}) \setminus \mathbf{P}(K \cdot x)$

The closure ordering on the set of K-orbits in $\mathcal{N}(\mathfrak{p})$ (resp., $\mathrm{Ad}(\mathfrak{g}_{\mathbb{R}})$-orbits in $\mathcal{N}(\mathfrak{g}_{\mathbb{R}})$) is defined by the condition that $\mathcal{O}_1 > \mathcal{O}_2$ if and only if $\mathcal{O}_2$ is contained in the closure of $\mathcal{O}_1$ and $\mathcal{O}_1 \neq \mathcal{O}_2$. According to [BS], the Kostant–Sekiguchi bijection preserves the closure ordering. Clearly its describing can be reduced to the case of simple $\mathfrak{g}$. In this case, the explicit description of the closure ordering is obtained in [D1], [D2], [D8], [D9], [D10], [D11], [D12], [D13] (see also [O], [Se]).

This information and Theorems 8–14 yield the orbit decomposition of every irreducible component of the orbit boundary $\mathbf{P}(\overline{K \cdot x}) \setminus \mathbf{P}(K \cdot x)$ for any (-1)-distinguished K-orbit $K \cdot x$. In particular, this gives the dimensions of these components and their intersection configurations.

Singular locus of a self-dual variety $\mathbf{P}(\overline{K \cdot x})$

Apart from a rather detailed information on the singular loci of the self-dual projectivized nilpotent orbit closures in the adjoint case (see [P2]), in general case only a partial information is available. In [O], for $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}_n(\mathbb{C}), \mathfrak{so}_n(\mathbb{C}))$ and $(\mathfrak{sl}_{2n}(\mathbb{C}), \mathfrak{sp}_n(\mathbb{C}))$, the normality of the nilpotent orbit closures $\overline{K \cdot x}$ (hence that of $\mathbf{P}(\overline{K \cdot x})$) is studied. In particular, it is shown there that for $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}_{2n}(\mathbb{C}), \mathfrak{sp}_n(\mathbb{C}))$, normality holds for any $\overline{K \cdot x}$, but if $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}_n(\mathbb{C}), \mathfrak{so}_n(\mathbb{C}))$, this is not the case. See also [Se], [O], [SeSh], where the local equations of the generic singularities of some orbit closures in $\mathbf{P}(\mathcal{N}(\mathfrak{p}))$ are found. By [P3, Proposition 4], Hesselink's desingularization of the closures of Hesselink strata, [He], cf. [PV], yields a desingularization of any orbit closure $\mathbf{P}(\overline{K \cdot x})$ in $\mathbf{P}(\mathcal{N}(\mathfrak{p}))$ (in [R], another approach to desingularization of such orbit closured is considered).

TABLE 7

(-1)-distinguished K-orbits $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$ for $\mathfrak{g}_{\mathbb{R}} = \mathtt{E}_{6(6)}$

No.	Dyn $K \cdot x$	Dyn $G \cdot x$	$\dim_{\mathbb{C}} K \cdot x$	$\sharp(G \cdot x \cap \mathfrak{p})/K$	$\mathfrak{z}_{\mathfrak{k}}(x)/\mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$	$\dim_{\mathbb{C}} \mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$
1	2222	202222	35	1	0	1
2	2202	220002	33	2	0	3
3	0220	220002	33	2	0	3
4	4224	222222	36	1	0	0

TABLE 8

(-1)-distinguished K-orbits $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$ for $\mathfrak{g}_{\mathbb{R}} = \mathtt{E}_{6(2)}$

No.	Dyn $K \cdot x$	Dyn $G \cdot x$	$\dim_{\mathbb{C}} K \cdot x$	$\sharp(G \cdot x \cap \mathfrak{p})/K$	$\mathfrak{z}_{\mathfrak{k}}(x)/\mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$	$\dim_{\mathbb{C}} \mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$
1	000004	020000	21	3	$2\mathtt{A}_2$	1
2	301000	001010	25	3	$\mathtt{A}_1 + \mathtt{T}_1$	9
3	001030	001010	25	3	$\mathtt{A}_1 + \mathtt{T}_1$	9
4	004000	000200	29	3	$\mathtt{T}_2$	7
5	020204	000200	29	3	$\mathtt{T}_2$	7
6	004008	020200	30	2	$\mathtt{A}_2$	0
7	400044	220002	30	2	$\mathtt{A}_1 + \mathtt{T}_1$	4
8	121131	111011	31	2	$\mathtt{T}_1$	6
9	311211	111011	31	2	$\mathtt{T}_1$	6
10	313104	121011	32	2	$\mathtt{T}_1$	5
11	013134	121011	32	2	$\mathtt{T}_1$	5
12	222222	200202	33	2	0	5
13	040404	200202	33	2	0	5
14	224224	220202	34	2	$\mathtt{T}_1$	3
15	404048	220202	34	2	$\mathtt{T}_1$	3
16	440444	222022	35	1	0	3
17	444448	222222	36	1	0	2

TABLE 9

(-1)-distinguished K-orbits $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$ for $\mathfrak{g}_{\mathbb{R}} = \mathtt{E}_{6(-26)}$

No.	Dyn $K \cdot x$	Dyn $G \cdot x$	$\dim_{\mathbb{C}} K \cdot x$	$\sharp(G \cdot x \cap \mathfrak{p})/K$	$\mathfrak{z}_{\mathfrak{k}}(x)/\mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$	$\dim_{\mathbb{C}} \mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$
1	0001	100001	16	1	$\mathtt{B}_3$	15
2	0002	200002	24	1	$\mathtt{G}_2$	14

TABLE 10

(-1)-distinguished K-orbits $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$ for $\mathfrak{g}_{\mathbb{R}} = \mathtt{E}_{6(-14)}$

No.	Dyn $K \cdot x$	Dyn $G \cdot x$	$\dim_{\mathbb{C}} K \cdot x$	$\sharp(G \cdot x \cap \mathfrak{p})/K$	$\mathfrak{z}_{\mathfrak{k}}(x)/\mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$	$\dim_{\mathbb{C}} \mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$
1	100001	100001	16	3	$\mathtt{B}_3 + \mathtt{T}_1$	8
2	10000−2	100001	16	3	$\mathtt{B}_3 + \mathtt{T}_1$	8
3	10101−2	110001	23	2	$\mathtt{A}_2 + \mathtt{T}_1$	14
4	11100−3	110001	23	2	$\mathtt{A}_2 + \mathtt{T}_1$	14
5	40000−2	200002	24	1	$\mathtt{G}_2$	8
6	03001−2	120001	26	2	$\mathtt{B}_2 + \mathtt{T}_1$	9
7	01003−6	120001	26	2	$\mathtt{B}_2 + \mathtt{T}_1$	9
8	02202−6	220002	30	1	$\mathtt{A}_1 + \mathtt{T}_1$	12

TABLE 11

(-1)-distinguished K-orbits $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$ for $\mathfrak{g}_{\mathbb{R}} = \mathtt{E}_{7(-5)}$

No.	Dyn $K \cdot x$	Dyn $G \cdot x$	$\dim_{\mathbb{C}} K \cdot x$	$\sharp(G \cdot x \cap \mathfrak{p})/K$	$\mathfrak{z}_{\mathfrak{k}}(x)/\mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$	$\dim_{\mathbb{C}} \mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$
1	0000004	2000000	33	3	$\mathtt{A}_5$	1
2	4000000	0000020	42	2	$\mathtt{G}_2 + \mathtt{A}_1$	10
3	0000400	0020000	47	3	$3\mathtt{A}_1$	13
4	0002004	0020000	47	3	$3\mathtt{A}_1$	13
5	0000408	2020000	48	2	$\mathtt{C}_3$	0
6	2010112	0001010	49	1	$\mathtt{A}_1 + \mathtt{T}_1$	16
7	0400004	2000020	50	2	$\mathtt{A}_2 + \mathtt{T}_1$	10
8	1111101	1001010	52	1	$\mathtt{T}_2$	15
9	2010314	2001010	53	1	$\mathtt{A}_1 + \mathtt{T}_1$	12
10	0040000	0002000	53	1	$\mathtt{A}_1$	13
11	0202202	0020020	55	2	$\mathtt{A}_1$	11
12	0004004	0020020	55	2	$\mathtt{A}_1$	11
13	0202404	2020020	56	2	$2\mathtt{A}_1$	7
14	0400408	2020020	56	2	$2\mathtt{A}_1$	7
15	4004000	0002020	57	1	$\mathtt{A}_1$	9
16	0404004	2002020	59	1	$\mathtt{T}_1$	9
17	0404408	2022020	60	1	$\mathtt{A}_1$	6

TABLE 12

(-1)-distinguished K-orbits $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$ for $\mathfrak{g}_{\mathbb{R}} = \mathtt{E}_{7(7)}$

No.	$\mathrm{Dyn}\, K \cdot x$	$\mathrm{Dyn}\, G \cdot x$	$\dim_{\mathbb{C}} K \cdot x$	$\sharp(G \cdot x \cap \mathfrak{p})/K$	$\mathfrak{z}_{\mathfrak{k}}(x)/\mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$	$\dim_{\mathbb{C}} \mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$
1	4000000	0200000	42	4	$\mathtt{G}_2$	7
2	0000004	0200000	42	4	$\mathtt{G}_2$	7
3	0040000	0000200	50	4	$\mathtt{A}_1$	10
4	0000400	0000200	50	4	$\mathtt{A}_1$	10
5	3101021	1001010	52	3	$\mathtt{T}_2$	9
6	1201013	1001010	52	3	$\mathtt{T}_2$	9
7	4004000	2000200	54	4	$\mathtt{A}_1$	6
8	0004004	2000200	54	4	$\mathtt{A}_1$	6
9	2220202	0002002	56	4	0	7
10	2020222	0002002	56	4	0	7
11	0400400	0002002	56	4	0	7
12	0040040	0002002	56	4	0	7
13	2222202	2002002	58	4	0	5
14	2022222	2002002	58	4	0	5
15	4004040	2002002	58	4	0	5
16	0404004	2002002	58	4	0	5
17	4220224	2002020	59	2	$\mathtt{T}_1$	3
18	2422222	2002022	60	4	0	3
19	2222242	2002022	60	4	0	3
20	4404040	2002022	60	4	0	3
21	0404044	2002022	60	4	0	3
22	4404404	2220202	61	2	0	2
23	4044044	2220202	61	2	0	2
24	4444044	2220222	62	2	0	1
25	4404444	2220222	62	2	0	1
26	8444444	2222222	63	2	0	0
27	4444448	2222222	63	2	0	0

TABLE 13

(-1)-distinguished K-orbits $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$ for $\mathfrak{g}_{\mathbb{R}} = \mathtt{E}_{7(-25)}$

No.	Dyn $K \cdot x$	Dyn $G \cdot x$	$\dim_{\mathbb{C}} K \cdot x$	$\sharp(G \cdot x \cap \mathfrak{p})/K$	$\mathfrak{z}_{\mathfrak{k}}(x)/\mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$	$\dim_{\mathbb{C}} \mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$
1	0000002	0000002	27	4	$\mathtt{F}_4$	0
2	000000−2	0000002	27	4	$\mathtt{F}_4$	0
3	010010−2	1000010	38	2	$\mathtt{A}_3 + \mathtt{T}_1$	25
4	011000−3	1000010	38	2	$\mathtt{A}_3 + \mathtt{T}_1$	25
5	200002−2	2000002	43	4	$\mathtt{B}_3$	15
6	400000−2	2000002	43	4	$\mathtt{B}_3$	15
7	000004−6	2000002	43	4	$\mathtt{B}_3$	15
8	200002−6	2000002	43	4	$\mathtt{B}_3$	15
9	220002−6	2000020	50	1	$\mathtt{A}_2 + \mathtt{T}_1$	20
10	400004−6	2000022	51	2	$\mathtt{G}_2$	14
11	400004−10	2000022	51	2	$\mathtt{G}_2$	14

TABLE 14

(-1)-distinguished K-orbits $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$ for $\mathfrak{g}_{\mathbb{R}} = \mathtt{E}_{8(8)}$

No.	Dyn $K \cdot x$	Dyn $G \cdot x$	$\dim_{\mathbb{C}} K \cdot x$	$\sharp(G \cdot x \cap \mathfrak{p})/K$	$\mathfrak{z}_{\mathfrak{k}}(x)/\mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$	$\dim_{\mathbb{C}} \mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$
1	40000000	20000000	78	3	$2\mathtt{G}_2$	14
2	00000004	02000000	92	3	$\mathtt{A}_2$	20
3	21010100	00010001	96	3	$\mathtt{A}_1 + \mathtt{T}_1$	20
4	00400000	00000200	97	2	$2\mathtt{A}_1$	17
5	40000040	02000002	99	3	$\mathtt{A}_2$	13
6	20200200	00002000	104	3	0	16
7	00004000	00002000	104	3	0	16
8	02002002	00002000	104	3	0	16
9	40040000	20000200	105	2	$2\mathtt{A}_1$	9
10	02002022	00002002	107	3	$\mathtt{T}_1$	12
11	00400040	00002002	107	3	$\mathtt{T}_1$	12
12	31010211	10010101	108	2	$\mathtt{T}_1$	11
13	13111101	10010102	109	2	$\mathtt{T}_1$	10
14	20202022	00020002	110	2	0	10
15	04004000	00020002	110	2	0	10
16	02022022	20002002	111	3	$\mathtt{T}_1$	8
17	40040040	20002002	111	3	$\mathtt{T}_1$	8

TABLE 14 (*continued*)

(-1)-distinguished K-orbits $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$ for $\mathfrak{g}_{\mathbb{R}} = \mathtt{E}_{8(8)}$

No.	Dyn $K \cdot x$	Dyn $G \cdot x$	$\dim_{\mathbb{C}} K \cdot x$	$\sharp(G \cdot x \cap \mathfrak{p})/K$	$\mathfrak{z}_{\mathfrak{k}}(x)/\mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$	$\dim_{\mathbb{C}} \mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$
18	00400400	00020020	112	2	0	8
19	22202022	00020020	112	2	0	8
20	22202042	00020022	113	2	0	7
21	04004040	00020022	113	2	0	7
22	22222022	20020020	114	2	0	6
23	40040400	20020020	114	2	0	6
24	22222042	20020022	115	2	0	5
25	04040044	20020022	115	2	0	5
26	22222222	20020202	116	2	0	4
27	44040400	20020202	116	2	0	4
28	24222242	20020222	117	2	0	3
29	44040440	20020222	117	2	0	3
30	44044044	22202022	118	1	0	2
31	44440444	22202222	119	1	0	1
32	84444444	22222222	120	1	0	0

TABLE 15

(-1)-distinguished K-orbits $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$ for $\mathfrak{g}_{\mathbb{R}} = \mathtt{E}_{8(-24)}$

No.	Dyn $K \cdot x$	Dyn $G \cdot x$	$\dim_{\mathbb{C}} K \cdot x$	$\sharp(G \cdot x \cap \mathfrak{p})/K$	$\mathfrak{z}_{\mathfrak{k}}(x)/\mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$	$\dim_{\mathbb{C}} \mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$
1	00000004	00000002	57	3	$\mathtt{E}_6$	1
2	00000204	00000020	83	3	$\mathtt{D}_4$	25
3	00000040	00000020	83	3	$\mathtt{D}_4$	25
4	00000048	00000022	84	2	$\mathtt{F}_4$	0
5	01100012	10000100	89	1	$\mathtt{B}_2 + \mathtt{T}_1$	36
6	40000004	20000002	90	2	$\mathtt{A}_4$	22
7	10100111	10000101	94	1	$\mathtt{A}_2 + \mathtt{T}_1$	33
8	01100034	10000102	95	1	$\mathtt{A}_3$	26
9	00020000	00000200	97	1	$2\mathtt{A}_1$	33
10	20000222	20000020	99	2	$\mathtt{G}_2$	23
11	00000404	20000020	99	2	$\mathtt{G}_2$	23
12	20000244	20000022	100	2	$\mathtt{B}_3$	15
13	40000048	20000022	100	2	$\mathtt{B}_3$	15

TABLE 15 (*continued*)

(-1)-distinguished K-orbits $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$ for $\mathfrak{g}_{\mathbb{R}} = \mathtt{E}_{8(-24)}$

No.	Dyn $K \cdot x$	Dyn $G \cdot x$	$\dim_{\mathbb{C}} K \cdot x$	$\sharp(G \cdot x \cap \mathfrak{p})/K$	$\mathfrak{z}_{\mathfrak{k}}(x)/\mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$	$\dim_{\mathbb{C}} \mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$
14	00020200	20000200	105	1	$2\mathtt{A}_1$	25
15	40000404	20000202	107	1	$\mathtt{A}_2$	21
16	40000448	20000222	108	1	$\mathtt{G}_2$	14

TABLE 16

(-1)-distinguished K-orbits $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$ for $\mathfrak{g}_{\mathbb{R}} = \mathtt{F}_{4(4)}$

No.	Dyn $K \cdot x$	Dyn $G \cdot x$	$\dim_{\mathbb{C}} K \cdot x$	$\sharp(G \cdot x \cap \mathfrak{p})/K$	$\mathfrak{z}_{\mathfrak{k}}(x)/\mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$	$\dim_{\mathbb{C}} \mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$
1	0004	2000	15	3	$\mathtt{A}_2$	1
2	0040	0200	20	3	0	4
3	0204	0200	20	3	0	4
4	2022	0200	20	3	0	4
5	2200	0048	21	2	$\mathtt{A}_1$	0
6	0404	0202	22	2	0	2
7	2222	0202	22	2	0	2
8	2244	2202	23	2	0	1
9	4048	2202	23	2	0	1
10	4448	2222	24	1	0	0

TABLE 17

(-1)-distinguished K-orbits $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$ for $\mathfrak{g}_{\mathbb{R}} = \mathtt{F}_{4(-20)}$

No.	Dyn $K \cdot x$	Dyn $G \cdot x$	$\dim_{\mathbb{C}} K \cdot x$	$\sharp(G \cdot x \cap \mathfrak{p})/K$	$\mathfrak{z}_{\mathfrak{k}}(x)/\mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$	$\dim_{\mathbb{C}} \mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$
1	0001	0001	11	1	$\mathtt{A}_3$	10
2	4000	0002	15	1	$\mathtt{G}_2$	7

TABLE 18

(-1)-distinguished K-orbits $K \cdot x$ in $\mathcal{N}(\mathfrak{p})$ for $\mathfrak{g}_{\mathbb{R}} = \mathtt{G}_{2(2)}$

No.	Dyn $K \cdot x$	Dyn $G \cdot x$	$\dim_{\mathbb{C}} K \cdot x$	$\sharp(G \cdot x \cap \mathfrak{p})/K$	$\mathfrak{z}_{\mathfrak{k}}(x)/\mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$	$\dim_{\mathbb{C}} \mathrm{rad}_u \mathfrak{z}_{\mathfrak{k}}(x)$
1	22	02	5	2	0	1
2	04	02	5	2	0	1
3	48	22	6	1	0	0

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